THE

Mathematical Correspondent;

CONTAINING,

NEW ELUCIDATIONS, DISCOVERIES, AND IMPROVEMENTS,

IN VARIOUS BRANCHES OF THE MATHEMATICS;

WITH

COLLECTIONS OF MATHEMATICAL QUESTIONS RESOLVED BY INGENIOUS CORRESPONDENTS.

ADAPTED

To the present state of learning in America.

AND DESIGNED,

TO INSPIRE YOUTH WITH THE LOVE OF MATHEMATICAL KNOWLEDGE, BY ALLURING THEIR ATTENTIONS TO THE SOLUTIONS OF PLEASANT AND CURIOUS QUESTIONS—AND TO PROMOTE THE CULTIVATION OF THE MATHEMATICS, BY OPENING A CHANNEL FOR THE READY CONVEYANCE OF DISCOVERIES AND IMPROVEMENTS, FROM ONE MATHEMATICIAN TO ANOTHER.

VOL. I.

"In the Mathematical Sciences, truth appears most conspicuous, and shines in its greatest lustre." EMERSON.

NEW-YORK:

PRINTED BY SAGE AND CLOUGH,

FOR THE EDITORS, AND SOLD BY T. AND J. SWORDS, NO. 160, W. FALCONER, NO. 112, SAGE AND THOMPSON, NO. 14, PEARL-STREET, AND C. LARON, NO. 79, PINE-STREET.

1804.

ASTOR LIBRARY NEW-YORK

Repair No. 336/03.

PREFACE.

WHEN we consider the great exertions of learned men to disseminate mathematical information in other countries, we must be surprised to find that this kind of knowledge is most shamefully neglected in the United States of America. The mathematical sciences are the foundation of almost every art that is necessary to promote the comfort and convenience of civilized man: their extensive use in human affairs stands attested by the wise and learned of every age. It cannot therefore be denied that these sciences ought, for the sake of their usefulness, to be diligently studied and liberally encouraged in a country like this. But these sciences are as valuable in their own nature as they are useful. For by mathematical exercise, as the celebrated Dr. Barrow observes, the mind is inured to a constant diligence in study, delivered from a credulous simplicity, strongly fortified against the vanity of scepticism, restrained from rash presumption, inclined to a due assent, subject to the government of right reason, and inspired with resolution to combat the unjust tyranny of false prejudices. In Europe small periodical publications have contributed much to the diffusion of this kind of learning, and many of the greatest scientific characters of the present age have at an early stage of life commenced their mathematical career by answering the questions proposed in such works. Ambitious of seeing their little productions in print, they studied with an ardour which nothing else could inspire; and this ambition properly encouraged, by a judicious parent or preceptor, accelerated the improvement of the pupil and formed the basis of future eminence. The learned Dr. Hutton in a late publication has declared that

a small periodical work, entitled, the "Ladies' Diary," had produced more mathematicians in England than had been done by all the mathematical authors in that kingdom. But this is not the only advantage that has resulted from such works. They have served as useful vehicles in conveying ideas, discoveries, and improvements, from one mathematician to another, and to the community at large. By these means the limits of science have been extended, and the common stock of mathematical knowledge continually augmented and enriched. The Mathematical Correspondent *will be conducted on the same plan as the European works to which we have just alluded; and as similar causes generally produce similar effects, we are not without hopes of rendering some service to the public.*

A number of this work containing one sheet of paper, will be regularly published four times a year, viz. on the first days of May, August, November, and February. In each number a prize question will be proposed, and whoever gives the best solution to that question, one month previous to the publication of the next succeeding number, shall receive a handsome silver medal on which is the following inscription, "From the Editors of the Mathematical Correspondent, to A—— B——, as a reward for his mathematical merit." *The small profits arising from the sale of the work will be applied to defraying the expence of the prize medals.*

With a view of rendering the work as useful as possible to the generality of our subscribers, we have begun with the lowest parts of the mathematics; and for this reason the questions proposed in the first number are such as may be resolved by a very small degree of Mathematical knowledge. In future, however, we shall gradually ascend towards the higher regions of those sciences, as far as may be thought consistent with the abilities of our readers. On this subject the opinion and advice of every American mathematician, will always be thankfully received and duly considered.

THE

MATHEMATICAL CORRESPONDENT,

&c.

ARTICLE I.

A New elucidation of the principles of the Rule of Proportion, in Arithmetic; applied to the resolution of practical questions, and to the invention of general rules for making in the neatest and shortest manner possible, many of the most useful calculations that daily occur in the counting-house. *By* G. BARON.

§ 1. Any number may be represented by as many dots placed in a straight line as that number contains units. Thus the number 7 is represented by seven dots placed in the annexed straight line A.

A . . . . . .

§ 2. A greater number is a multiple of a less, when the greater contains the less a certain number of times exactly. Thus the number 12 represented by B is a multiple of 4 represented by C; for 12 contains 4 a certain number of times exactly (viz. 3 times). For the same reason 14 is a multiple of 7, 18 is a multiple of 6, &c. &c.

B . . . . | . . . . | . . . .

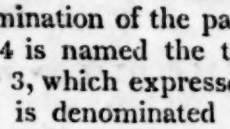
C . . . .

§ 3. When a greater number is a multiple of a less, the less is said to be a part of the greater. Thus the number 4 is a part of (its multiple) 12, 7 is a part of 14, 6 is a part of 18, &c. &c.

§ 4. The denominator of a part of any number, is the number which expresses how often that number

contains the part. It is called the denominator, because the name or denomination of the part is always derived from it. Thus 4 is named the third part of 12 from the denominator 3, which expresses how often 12 contains its part 4, 2 is denominated the seventh part of 14 from the denominator 7, which expresses how often 14 contains its part 2, &c. &c.

§ 5. When a number B is contained in one of its multiples A, as often as another number D is contained in one of its multiples C; then C is said to be the like multiple of D that A is of B; and D the like part of C that B is of A. Thus 9 is a multiple of 3, and 12 is a like multiple of 4; 3 is a part of 9, and 4 is a like part of 12. Again 18 is a multiple of 2, and 27 is a like multiple of 3; 2 is a part of 18, and 3 is a like part of 27, &c. &c.



§ 6. When four numbers have the properties expressed in any one of the three following cases, they are proportional; and when four numbers are proportional they have the properties contained in some one of the same cases.*

CASE I.

When the first number is a multiple of the second, and the third is a like multiple of the fourth.

ILLUSTRATION.

The four numbers 6, 2, 12, and 4, are proportional, for the first is a multiple of the second, and the third is a like multiple of the fourth. Four proportional numbers are universally written thus, 6 : 2 :: 12 : 4;

* This section must be considered, not as a definition, but merely as a description of the three properties, by which proportional numbers are commonly known. These three properties are actually three different propositions deduced from the true definition of proportional numbers. This definition is reserved as a curious subject for a future enquiry.

and read thus, 6 is to 2 as 12 is to 4. For the same reason 9 : 3 : : 15 : 5, 22 : 2 : : 33 : 3, &c. &c.

Case II.

When the first number is a part of the second, and the third is a like part of the fourth.

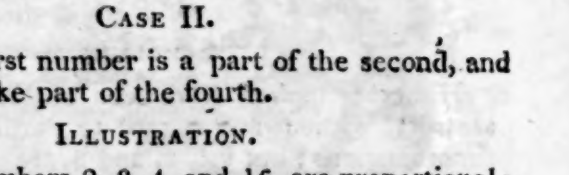
Illustration.

The four numbers 2, 8, 4, and 16, are proportional; for the first is a part of the second, and the third is a like part of the fourth. For the same reason, 3 : 15 : : 5 : 25, 11 : 44 : : 9 : 36, &c. &c.

Case III.

When the first number is a multiple of a part of the second, and the third is a like multiple of a like part of the fourth.

Illustration.



The four numbers, 6, 8, 9, and 12, represented by A, B, C, and D are proportional. For E is a part of the second B, and F is a like part of the fourth D; and A is a multiple of (E) a part of B, and C is a like multiple of (F) a like part of D. Consequently A : B : : C : D.

In the same manner it may be shewn that 9 : 15 : : 12 : 20, that 21 : 14 : : 36 : 24, that &c. &c.

When A, B, C, and D, represent any four proportional numbers, the first and fourth, A and D, are called the extremes, and the second and third, B and C, the means. The last number, D, is said to be a fourth proportional to A, B, and C; and the numbers A, B, C, and D, are denominated the proportional terms.

Corollary.

When A : B : : C : D, according to case III. of § 6;

B is a multiple of (E) a part of A, and D is a like multiple of (F) a like part of C. This follows from § 5; and hence B : A :: D : C; (by case III. § 6.)

Scholium.

Hitherto numbers have been considered as pure or abstract, without any regard to particular kinds and denominations; but § 1, 2, and 3, are also true of numbers of the same particular kinds. For, in § 1, 7 dollars may be represented by 7 dots. In § 2 and 3, 12 crowns are a multiple of 3 crowns, 14 cents are a multiple of 7 cents; 3 crowns are a part of 12 crowns, 7 cents are a part of 14 cents, &c. &c. But a number of one kind cannot be either a multiple or a part of another number of a different kind; for 12 years cannot be a multiple of 3 inches, nor can 7 oxen be a part of 21 sheep.

In § 4, the denominator of a part is a pure number, and can never be of any particular kind whatever. For although 4 men be a third part of 12 men, yet the denominator 3 is a pure number, independent of every particular kind or denomination.

In the rule of multiplication the product is a multiple of the multiplicand, consequently the multiplicand is a part of the product, and the multiplier is the denominator of that part. The product is therefore always of the same kind with the multiplicand, and the multiplier is always a pure number.* Hence a number of one kind cannot be multiplied by a number either of the same or of a different kind: 3 *l.* cannot be multiplied by 5 *l.* 7 *feet* cannot be multiplied by 4 feet; nor can 6 bears be multiplied by 9 asses. All such questions are evidently unscientific and absurd, and serve only to demonstrate the ignorance and stupidity of their authors.

In the rule of division, either a number and one of its parts are given to find the denominator of that part;

* See § 4, chap. 7, book 1, Malcolm's Arith. page 80 and 134 Fenn's Arith. and art. 10 of Baron Maseres's Dissertation on the use of the Negative Sign.

or a number and the denominator of one of its parts are given to find the part. Hence if the divisor and dividend are numbers of the same kind, the quotient must be a pure number; but if the divisor be a pure number, the quotient is of the same kind with the dividend. Let 3 yards be the divisor, and 24 yards the dividend, then the quotient or denominator 8 is a pure number, which indicates that 3 yards are the eighth part of 24 yards. Again, the sixth part of 30 dollars is 5 dollars: here the denominator 6, a pure number, is the divisor, and the quotient 5 dollars is of the same kind with the dividend 30 dollars.

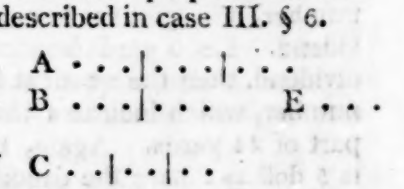
In § 5, one number may be either a multiple or a part of a second number of the same kind; and a third number may be a like multiple or a like part of a fourth, of the same kind, with regard to each other, although of a different kind from that of the two former. Thus 8 yards are a multiple of 2 yards, and 12 dollars are a like multiple of 3 dollars; 2 yards are a part of 8 yards, and 3 dollars are a like part of 12 dollars. The same is also true if the pure numbers 8 and 2 be taken instead of 8 yards and 2 yards.

From § 6, and what has already been shewn in this scholium, it evidently follows, that in four proportional numbers, the first and second must always be of the same kind; and the third and fourth also of one kind, notwithstanding the two latter may be of a kind different from that of the two former. And hence the ridiculous absurdity of the customary manner of stating questions in the rule of proportion, by making the first number of one kind and the second of a different kind; is abundantly manifest.

§ 7. In four proportional numbers, if the first be greater than the second, the third is greater than the fourth; but if the first be less than the second, the third is less than the fourth.

This is evidently true, when four proportional numbers have one of the properties described in the first

and second cases of § 6. For any multiple of a number must be greater, and any part must be less, than the number itself. It only remains therefore to prove that the same is also true when four proportional numbers have the properties described in case III. § 6.



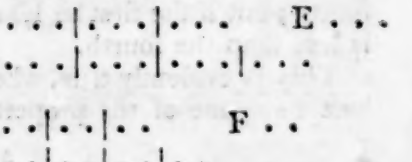
Let A, B, C, and D, represent any four proportional numbers, in which A is a multiple of E, a part of B; and C is a like multiple of F, a like part of D. Let the first number, A, be greater than the second, B; then shall the third, C, be greater than the fourth, D. For since E is a part of B, and F is a like part of D; B contains E exactly as often as D contains F (by § 5). And since A is a multiple of E, and C is a like multiple of F; A contains E exactly as often as C contains F (by § 5). But since A is greater than B, A must contain E oftener than B contains E; and consequently C must contain F oftener than D contains F; and therefore C is greater than D.

In the same manner it may be proved that when B is greater than A, D is greater than C; and consequently when A is less than B, C is less than D.

COROLLARY.

If the first of four proportional numbers be equal to the second; the third is equal to the fourth. This naturally follows from the demonstration of this section.

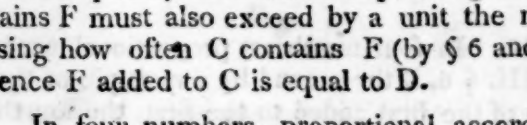
§ 8. In four numbers, proportional according to case III. § 6, if a part of the first, added to the first, be equal to the second; a like part of the third, added to the third, will be equal to the fourth.



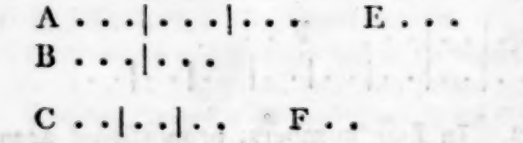
For when A : B :: C : D according to case III. § 6; B is a multiple of (E) a part of A, and D is a like multiple of

(F) a like part of C (by cor. to § 6). When therefore E, a part of A, added to A is equal to B; the number which expresses how often B contains E, must exceed by a unit, the number expressing how often A contains E; and consequently the number expressing how often D contains F must also exceed by a unit the number expressing how often C contains F (by § 6 and cor.) And hence F added to C is equal to D.

§ 9. In four numbers, proportional according to case III. § 6, if a part of the first subtracted from the first leave a remainder equal to the second; a like part of the third subtracted from the third will leave a remainder equal to the fourth.*

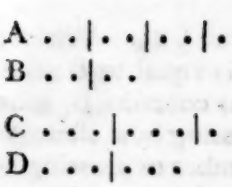


§ 10. In four numbers, proportional according to case III. § 6, if a part of the first added to the first be a multiple of the second; a like part of the third added to the third will be a like multiple of the fourth.

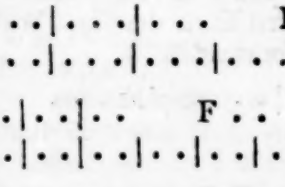


§ 11. In four numbers, proportional according to case III. § 6, if a part of the first, subtracted from the first, leave a remainder equal to a multiple of the second; a like part of the third, subtracted from the third, will leave a remainder equal to a like multiple of the fourth.

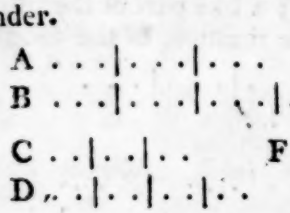
* As this and the next five fucceeding fections are almoft felf-evidently true; the confined limits of this work have induced us to omit their demonftrations. The demonftration of each of thefe fections is however founded on the fame principles with that of § 8, and may be eafily formed by the reader.



§ 12. In four numbers, proportional according to case III. § 6, if the second be a multiple of the sum of a part of the first added to the first, the fourth will be a like multiple of the sum of a like part of the third added to the third.



§ 13. In four numbers, proportional according to case III, § 6, if a part of the first be subtracted from the first, and the second be a multiple of the remainder; then if a like part of the third be taken from the third, the fourth will be a like multiple of this last remainder.



§ 14. If M and N represent any two pure numbers, then M × N is equal to N × M.

For at equal distances, let as many dots be placed in a horizontal straight line AB, as the number M contains units; and directly under this line, let the same number of dots be repeated in other horizontal straight lines, as often as there are units in the other number N, so as to form the rectangular figure ABCD. Then from the

nature of multiplication, it is evident by inspection that the number of dots in ABCD is the product of M × N. But as the vertical straight line AD must contain as many dots as there are units in the other number N; and as the same number of dots in AD, repeated in other vertical straight lines, as often as the number M contains units, actually constitutes the rectangle ABCD; the number of dots in ABCD is also equal to the product of N × M. Hence M × N is equal to N × M.

Corollary 1.

Hence M × N is a multiple of M, and also of N; and M and N are parts of the same M × N.

Corollary 2.

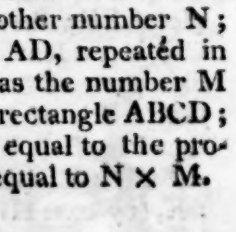
Hence any multiple of any number may be represented as above, by dots placed in a rectangular form.

§ 15. In four proportional numbers, any multiple of the first is to a like multiple of the second, as the third is to the fourth.

Case I.

When the first number AB is a multiple of the second EF, and the third I is a like multiple of the fourth K.

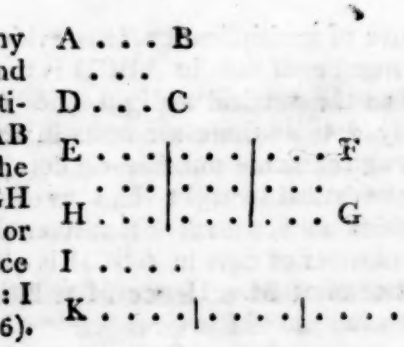
Let ABCD be any multiple of AB, and EFGH a like multiple of EF; then it is evident by inspection that ABCD is the like multiple of EFGH that AB is of EF, or that I is of K. Hence ABCD : EFGH :: I : K (by case I. § 6.)



Case II.

When the first AB is a part of the second EF, and the third I is a like part of the fourth K.

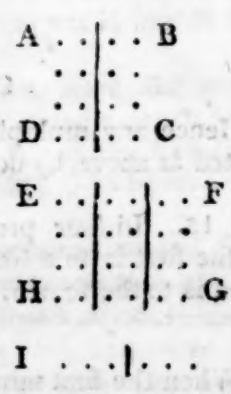
Let ABCD be any multiple of AB, and EFGH a like multiple of EF; then AB CD is evidently the like part of EFGH that AB is of EF, or that I is of K. Hence ABCD : EFGH :: I : K, (by case II. § 6).



Case III.

When the first AB is a multiple of a part of EF the second; and the third I is a like multiple of a like part of K the fourth.

Let ABCD be any multiple of AB, and EFGH a like multiple of EF; then it is evident by inspection that ABCD is a multiple of a part of EFGH, and that I is a like multiple of a like part of K. Hence ABCD : EFGH :: I : K (by case III. § 6.)



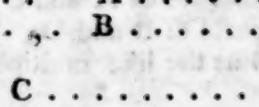
Corollary 1.

What has been demonstrated in this section of the like multiples of the first and second, is also true of like multiples of the third and fourth: for the same reasoning is equally applicable in both cases.

Corollary 2.

In any four numbers A, B, C, and D, if any multiple of A be to a like multiple of B as C is to D, then A : B :: C : D. This is evident from the demonstration of the three cases in this section.

§ 16. If A : B :: C : D, then any part of A is to a like part of B as C is to D. For



let E be any part of A, and F a like part of B; then A is a multiple of E, and B is a like multiple of F (by § 5). Now since A (a multiple of E) is to B (a like multiple of F) as C is to D; therefore E : F :: C : D (by cor. 2, § 15).

COROLLARY.

What has been demonstrated in cor. 2, § 15 and § 16, of the like parts of the first and second, of four proportional numbers, may exactly in the same manner be proved of the like parts of the third and fourth.

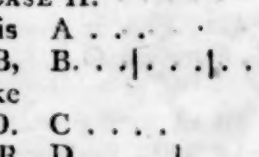
§ 17. If four numbers be proportional, any multiple of the first is to the second, as a like multiple of the third is to the fourth.

CASE I.

When the first AD is a multiple of the second B, and the third CG is a like multiple of the fourth K. Let ADEF be a multiple of AD, and CGHI a like multiple of CG; then it is evident by inspection that ADEF is a multiple of B, and that CGHI is a like multiple of K. Hence ADEF : B :: CGHI : K (by case I. § 6.)

CASE II.

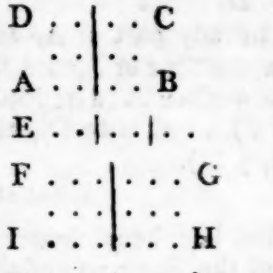
When the first A is a part of the second B, and the third C is a like part of the fourth D. Since A is a part of B and C is a like part of D,



any like multiples of A and C must be like multiples of the like parts of B and D. Therefore any multiple of A is to B as the like multiple of C is to D (by case III. § 6).

CASE III.

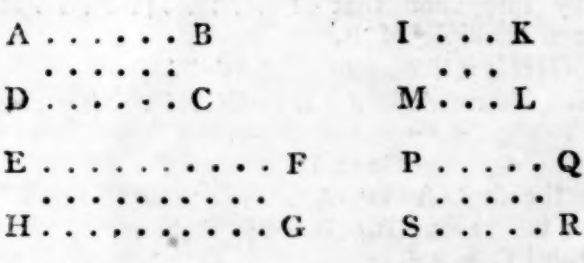
When the first AB is a multiple of a part of E the second, and the third FG is a like multiple of a like part of K the fourth. Let ABCD be any multiple of AB, and FGHI a like multiple of FG; then it is easy to perceive that ABCD is a multiple of a part of E, and that FGHI is a like multiple of a like part of K. Consequently ABCD : E :: FGHI : K (by case III. § 6.)



COROLLARY.

What has been demonstrated in this section, concerning the like multiples of the first and third numbers, is also true of the like multiples of the second and fourth: for the same reasoning is equally applicable to both.

§ 18. If four numbers be proportional, any part of the first is to the second, as a like part of the third is to the fourth.



Let the first of four proportional numbers be represented by ABCD, the second by EF, the third

by IKLM, and the fourth by PQ. Let AB be a part of ABCD, and IK a like part of IKLM; then ABCD is a multiple of AB, and IKLM is a like multiple of IK (by § 5.) Make EFGH the like multiple of EF, that ABCD is of AB (or that IKLM is of IK), and PQRS the like multiple of PQ. Then since ABCD : EF :: IKLM : PQ; therefore ABCD : EFGH::IKLM:PQRS (by cor. to § 17). But AB is the like part of ABCD, that EF is of EFGH, (that IK is of IKLM), and that PQ is of PQRS (by const. and § 5); therefore AB : EF :: IKLM : PQRS (by § 16). And again AB :: EF :: IK : PQ (by cor. to § 16); or AB a part of the first, is to the second EF, as IK a like part of the third, is to the fourth PQ.

COROLLARY.

What is here demonstrated, of the like parts of the first and third, is also true of the like parts of the second and fourth; for the same reasoning may be applied in both cases.

§ 19. If any four numbers be proportional, the product of the extremes is equal to the product of the means.

For let A, B, C and D represent any four proportional numbers; then since A : B :: C : D; A × B : B :: C × B : D (by cor. 1 § 14 and § 17). Again A × B : B × A :: C × B : D × A (by cor. 1 § 14 and cor. to § 17). But A × B is equal to B × A (by § 14); therefore C × B is equal to D × A (by cor. to § 7). Hence the product of the extremes A and D, is equal to the product of the means B and C.

COROLLARY.

Hence in four proportional numbers, if the product of the means be divided by one of the extremes, the quotient will be equal to the other extreme; for if A : B :: C : D then by this section D × A = C × B, and because the like parts of equal numbers must be equal one to another, D = C × B ÷ A: and there-

fore a fourth proportional to three given numbers may be found by multiplying the second and third terms together and dividing their product by the first.

SCHOLIUM.

In this section the first and second of the four proportional terms must always be considered as pure or abstract numbers: this follows from the demonstration of this section, and the scholium to § 6. Whoever considers the nature of proportional numbers will readily perceive that though the first and second terms must always be of the same kind in respect to each other, yet they are not limited to any particular kind, and ought therefore to be esteemed as pure numbers. For example—3 days : 5 days :: 6 yards : 10 yards, 3 men : 5 men :: 6 yards : 10 yards, 3 ounces : 5 ounces :: 6 yards : 10 yards, &c. &c; and the same is true of the first and second terms of any four proportional numbers: all this is evident from § 6, and hence the pure number 3 : the pure number 5 :: 6 yards : 10 yards.

It is also necessary to observe here, that when the first and second terms consist of different denominations, they must be reduced to the same denomination before they are considered as pure numbers: thus, if 2 feet 3 inches : 6 feet 9 inches :: 7 dollars : 21 dollars, the first term is 27 inches and the second is 81 inches; consequently the pure number 27 : the pure number 81 :: 7 dollars : 21 dollars.

GENERAL SCHOLIUM.

§ 20. The preceding sections have been limited to numbers for the sake of brevity; but several of these sections are equally true when some one of the numbers is a unit: this will evidently appear from the following remarks. In § 1 a unit may be represented by a single dot. In § 2 any number may be considered as a multiple of a unit. In § 3 a unit may be a part of any number, and the number itself is the de-

nominator of that part. In § 5 one number may be a multiple of a unit, and a second number may be a like multiple of some other number;—one number may be a part of another, and a unit may be a like part of a third number.*

Having finished the elucidation of the principles of the rule of proportion in arithmetic; I shall now apply these principles, to the resolution of practical questions, and to the invention of general rules for making, in the neatest and shortest manner possible, many of the most useful calculations that daily occur in the counting-house.

TO BE CONTINUED.

* The theory contained in the foregoing sections extends no farther than is necessary for establishing the elements of the rule of proportion. The scientific reader, desirous of acquiring an extensive knowledge of the doctrine of proportional numbers, is recommended to a careful perusal of the seventh, eighth, and ninth books of Euclid, and Emerson's doctrine of Proportion.

ARTICLE II.

A PROBLEM, *by* Wm. Elliot, printer, *N. York.*

Given $a : b :: c^2 : d^2$, to find x in terms of a and b, so that $a + x : b + x :: c : d$.

Since $a : b :: c^2 : d^2$, it is well known that $\surd a : \surd b :: c : d$, and hence by the problem $a + x : b + x :: \surd a : \surd b$; from this analogy $a\surd b + x\surd b = b\surd a + x\surd a$, and therefore $x = \frac{b\surd a - a\surd b}{\surd b - \surd a} = \surd a\,b$ = a mean proportional between a and b.

Corollary.

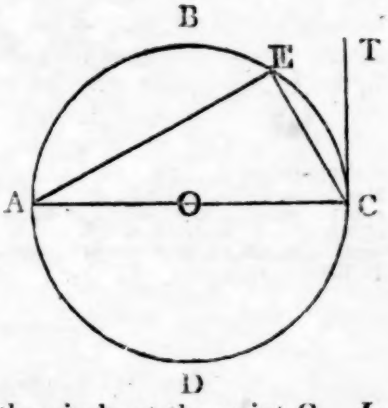
Hence in the Appollonian Parabola a number of curious properties become self-evident.

ARTICLE III.

A Curious and useful proposition in Geometry, with its application to a certain subject in Geography and Navigation. By R. TAGART.

PROPOSITION.

The radius of a circle intersects the circumference at right angles.



Let ABCD be a circle in which O is the centre and AOC the diameter, then shall the radius OC intersect the circumference at right angles in the point C.

From the point C draw CT at right angles to the radius OC, then by the 16th prop. of the 3d book of Euclid CT is a tangent to the circle at the point C. In the circumference of the circle take any point E and draw the straight lines CE and AE; then by the 2d prop. of the 3d book of Euclid, the straight line CE is entirely within the circle; and consequently the angle OCB formed by the radius and the circumference, cannot be less than the rectilineal angle OCE. But as the tangent CT is entirely without the circle, the same angle OCB cannot be greater than the rectilineal angle OCT. Hence then if the point E be taken any where on the circumference of the circle, the following conclusion is universally true, viz. *The angle* OCB *formed by the radius and the circumference is neither less than the rectilineal angle* OCE, *nor greater than the rectilineal angle* OCT. Again, by the 32d prop. of the 3d book of Euclid, the rectilineal angle ECT is equal to the angle EAC in the alternate segment. Imagine, now, the point E to move along the circumfer-

ence of the circle towards C, then will the angle EAC, during that motion, continually decrease. And because the angle ECT is always equal to the angle EAC, the angle ECT must also continually decrease; and consequently the straight line CE must revolve about the point C and continually approximate nearer to CT, during the motion of the point E towards C. But when the point E actually arrives at C, the straight line AE will coincide with AC, and the angles EAC and ECT will become nothing: the straight line EC will then coincide with CT, and the rectilineal angle OCE will likewise coincide and become equal to the angle OCT. Therefore, since OCT is a right angle by construction, OCE is also a right angle when the point E arrives at C. Hence then it must follow from the general conclusion above drawn, that the angle OCB formed by the radius and circumference of the circle; is neither less nor greater than a right angle: and therefore the radius of a circle intersects the circumference at right angles. Q. E. D.

Corollary.

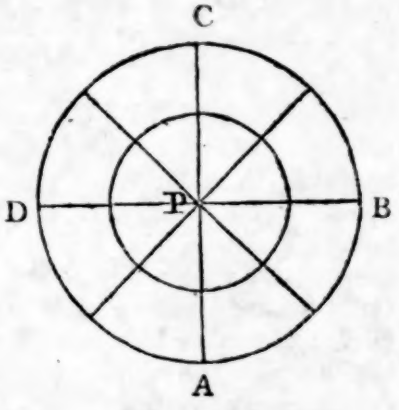
"An angle is the inclination of two lines to one "another, that meet together but are not in the same "direction." But by this proposition the line forming the circular arc meets its tangent in the same direction; and consequently the circular arc makes no angle with its tangent. Here then we see that two lines may meet together without forming an angle; and that the modern mysteries concerning the infinite divisibility of the angle of contact between a circular arc and its tangent, have all originated in the supposed existence of a thing which in reality does not and cannot exist.

Scholium.

By the help of this proposition it may be easily proved that the meridian circles on the globular earth in-

tersect the parallels of latitude at right angles, and consequently that a ship steered either due east or west must sail on a parallel of latitude.

Let a hemisphere of the earth be projected stereographically on the plane of the equatorial circle ABCD. Then it is well known that the pole will be projected into P the centre of the equator or primitive circle AB CD — that the meridian circles will be represented by straight lines PA, PB, PC, PD, &c. which are radii to the primitive—and that the parallels of latitude will be projected into smaller circles concentric with or parallel to this primitive circle. Therefore since P is the centre of every parallel of latitude on the plane of projection, the straight lines PA, PB, PC, PD, &c. here representing the meridians, must intersect at right angles every circle which represents a parallel of latitude: this evidently follows from the preceding proposition. But it is demonstrated in the fourth proposition of the learned Mr. Emerson's Stereographic projection of the sphere, that "the angle "made by two circles on the surface of the sphere, is "equal to that made by their representatives upon the "plane of projection"; and therefore as it has already been shewn that on the plane of projection the representative of any meridian cuts the representative of any parallel of latitude at right angles, it follows that on the surface of the spherical earth every meridian intersects every parallel of latitude at right angles. And hence a ship steered either due east or west, must sail on a parallel of latitude. It will be recollected that



John Fraser, a teacher of navigation in Newport, Rhode-Island, and another person under the signature Loxodromicus, have in the public newspapers lately made various attempts to persuade the people of these states, " that a ship steering a due east or west course " does not sail on a parallel of latitude, but on a cer- " tain spiral line which would ultimately carry her " to the equator" : They might just as well have said that a due east or west course will carry a ship to the moon.

ARTICLE IV.

NEW QUESTIONS, to be answered in the next number.

I. QUESTION, *by William Scott, A. M. New-York.*

IT is required to establish a general rule for finding in the shortest manner possible, how much New-York currency is equal to any given sum of sterling money, suppose 657*l.* 18*s.* 9*d.*

II. QUEST. *by W. Scott, A. M.*

It is required to establish a general rule for finding in the shortest manner possible, how much sterling money is equal to any given sum of New-York currency, suppose 976*l.* 10*s.* 8*d.*

III. QUEST. *by John Boyd, of New-York.*

My age in years is a number consisting of two digits, $\frac{1}{7}$ of this number is a mean proportional between these two digits, and two years hence, my age will be a third proportional to the same two digits, directly as they stand in my present age ; How long will it be before I am sixty years old ?

IV. QUEST. *by R. Tagart, New-York.*

In baking a hemispherical loaf of bread of 10 inches radius, the crust was every where of an equal thickness, and the solidity of the crust was equal to half the solid content of the whole loaf ; What were the dimensions of the interior soft part ?

V. QUEST. *by Diarius Yankee, Bunker's-Hill.*

A wretch, who spurn'd the virtuous path below,
To heaven above by other means would go:
He by mechanic art a ladder rais'd,
Its height the vulgar and the learn'd amaz'd.
Thus Jacob did, said he, and why not I
Scale heaven as well as he? at least I'll try.
He mounts his ladder, rapidly ascends,
And bids adieu to all terrestrial friends---
Now wrapp'd in clouds unpierc'd by human eyes---
And now environ'd with the azure skies---
But now a ball red-hot he lets fall down,
Which fires a cannon at the base, whose sound
Ascends his ear, in thrice the length of time
The ball was falling from this height sublime.
Ye sages tell if he to heaven has got,
Or how far soaring from this earthly spot?

VI. QUEST. *by Walter Folger, jun. Nantucket.*

A gentleman has bought a rectangular piece of land whose perimeter is to be 100 rods; and he is to pay 1 dollar for each rod in the length of the land, and 3 dollars for each rod in the breadth. It is required to determine the length and breadth so that the quantity of land may be had at the cheapest rate possible.

VII. PRIZE QUEST. *by George Baron, New-York.*

In surveying a field in the form of an oblique angled rectilineal triangle, the first or least side measured 24 chains, the second or next greater side measured 37·44 chains; on the third or greatest side grew two large hemlock trees whose distance asunder was 16·80 chains; a straight line drawn from the obtuse angle to the tree nearest the first side, was perpendicular to the second side, and another straight line drawn from the same angle to the other tree, was perpendicular to the first side. Required the third side, the distance of each tree from the obtuse angle, and the area of the field.

ARTICLE V.

ANSWERS to the Questions proposed in Article IV.

I. QUEST. *answered by Wm. Green, New-York.*

SINCE 9 sterling sixpences are equal to a dollar, or 16 sixpences New-York currency ; it is evident that 9 is to 16 as any sum of sterling money is to its equivalent in New-York currency.

Hence 9 : 16 :: 657*l.* 18*s.* 9*d.* : the sum required, and 18 : 16 :: 1315*l.* 17*s.* 6*d.* : the sum required (by art. 1. § 17. Math. Corr.). Here $\frac{1}{9}$ of the first term subtracted from the first, leaves a remainder equal to the second ; and consequently $\frac{1}{9}$ of the third term subtracted from the third will leave a remainder equal to the fourth (by art. 1. § 9. Math. Corr.). The general rule in question is therefore as follows : *Multiply any sum of sterling money by* 2, *and* $\frac{1}{9}$ *of the product, subtracted from the product, leaves the New-York currency required.* Hence,

	l.	*s.*	*d.*	
	657	18	9	sterling.
			2	
9)	1315	17	6	
	146	4	2	
Answer	1169	13	4	New-York currency.

II. QUEST. *answered by Samuel Moor, N. York.*

From the reasons advanced in the preceding solution, we have 16 : 9 :: 976*l.* 10*s.* 8*d.* : the sum required, and 16 : 18 :: 976*l.* 10*s.* 8*d.* : twice the sum required (by art. 1. cor. to § 17. Math. Corr.). But here $\frac{1}{8}$ of the first term added to the first, is equal to the second ; and of consequence $\frac{1}{8}$ of the third term,

added to the third will be equal to the fourth (by art. 1. § 8. Math. Corr.). The following is therefore the general rule sought, viz. *The eighth part of any sum of New-York currency, added to that sum, is equal to twice the sterling money required.* Hence,

	l.	*s.*	*d.*	
8)	976	10	8	New-York currency.
	122	1	4	
2)	1098	12	0	
Answer	549	6	0	sterling.

III. QUEST. *answered by William Lenhart, York-Town, Pennsylvania.*

Let x = the digit in the ten's place, and xy^2 = the digit in the unit's, then it is evident from the question that y is greater than 1 but not greater than 3, and that $10x + xy^2$ = the age of the proposer. By the second condition of the quest. $7xy = 10x + xy^2$, or $7y - y^2 = 10$, and each side of this equation subtracted from $\frac{49}{4}$ leaves $\frac{49}{4} - 7y + y^2 = \frac{9}{4}$ the square roots of which (since y is not greater than 3) give $\frac{7}{2} - y = \frac{3}{2}$, whence $y = 2$. The two digits originally expressed by x and xy^2, will now become x and $4x$ and the age of the proposer $10x + xy^2$ will be $= 14x$. Again by the third condition of the quest. $x : 4x :: 4x : 16x = 14x + 2$, and hence $x = 1$, $xy^2 = 4$, and 14 years is the age of our young mathematical friend, who, in 46 years will attain the age of 60.

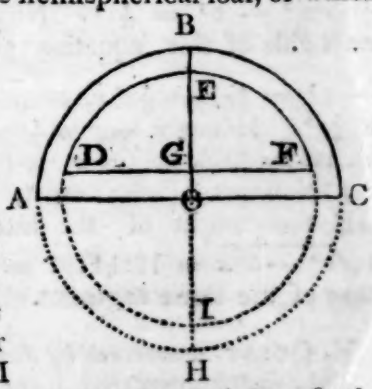
The same answered by R. Tagart, N. York.

Put x = the proposer's age, y = the digit in the ten's place and $x - 10y$ will be the digit in the place of units. Then by the second condition of the quest. $x = 7\sqrt{x - 10y \times y}$, or $x^2 = 49xy - 490y^2$.

Further, by the third condition of the quest. $y : x - 10y :: x - 10y : \frac{x^2 - 20xy + 100y^2}{y} = x + 2$, or $x^2 = 21xy - 100y^2 + 2y$, and this equated with the former value of x^2 gives $49xy - 490y^2 = 21xy - 100y^2 + 2y$, whence $x = \frac{195y + 1}{14}$, which by the quest. is a *whole number*. But since per quest. y, and of consequence $\frac{196y}{14}$ are also whole numbers; therefore $\frac{196y}{14} - \frac{195y + 1}{14} = \frac{y - 1}{14}$ must either be a whole number or 0, which being assumed $= p$ gives $y = 14p + 1$. Here it is evident that the digit y has but one possible value, that p must be $= 0$, and that $y = 1$; and hence $x = \frac{195y + 1}{14} = 14$ years, the age of Mr. Boyd, and $60 - 14 = 46$ years the time required.

IV. QUEST. *answered by Robert Adrain, York-Town, Pennsylvania.*

Let ABC represent the hemispherical loaf, of which O is the centre; then it is evident from the question that DEF, the interior soft part, is the segment of a sphere (DEFI) concentric with ABC the given hemisphere. Put $AO = OC = r = 10$ the given radius, $c = 3{\cdot}1416$, and $x = EI$ the diameter of the interior sphere DEFI, then is $GI = OH = r$ and $GE = x - r$ the height of the spherical segment DEF. Now by mensuration the



solid content of this segment is $\overline{x-r}^{2} \times \overline{x+2r} \times \frac{1}{6}c$ and the solid content of ABC the given hemisphere is $\frac{2}{3}cr^3$. Therefore by the question $\overline{x-r}^{2} \times \overline{x+2r} \times \frac{1}{6}c = \frac{1}{3}cr^3$, which reduced gives $x^2 = 3r^2$: and hence $x = r\sqrt{3} = 10\sqrt{3} = 17{\cdot}3205$ inches = EI and $x - r = 10 \times (\sqrt{3} - 1) = 7{\cdot}3205$ inches = GE, the dimensions required.

The same answered by William Thompson, Charleston, South-Carolina.

Put OA = OB = $r = 10$ the radius of the hemispherical loaf ABC, $c = 3{\cdot}1416$ and x = OG = BE the thickness of the crust; then will $r - x$ = OE the radius of the sphere DEFI of which DEF the interior soft part of the loaf is evidently a segment ; $r - 2x$ = GE the height of this segment and $2\sqrt{r^2 - 2rx}$ = DF the diameter of its base. Further, by mensuration $\overline{r-2x}^{2} \times \overline{2r-x} \times \frac{1}{3}c$ = solidity of the same segment and $\frac{2}{3}cr^3$ = solid content of the hemispherical loaf ABC : and therefore by the question $\overline{r-2x}^{2} \times \overline{2r-x} = r^3$, or $\frac{1}{2}r^3 - \frac{9}{4}r^2x + 3rx^2 - x^3 = \frac{1}{4}r^3$. Now $\frac{1}{2}r^3 - \frac{3}{4}r^2x$ added to each side of this equation gives $r^3 - 3r^2x + 3rx^2 - x^3 = \frac{3}{4}r^3 - \frac{3}{4}r^2x$, or $\overline{r-x}^{2} \times \overline{r-x} = \overline{r-x} \times \frac{3}{4}r^2$. Hence $r - x = \frac{1}{2}r\sqrt{3} = 5\sqrt{3} = 8{\cdot}660254$ inches = OE, $x = r \times (1 - \frac{1}{2}\sqrt{3}) = 1{\cdot}339746$ the thickness of the crust, $r - 2x = 7{\cdot}320508$ = GE the height of the interior soft segment and $2\sqrt{r^2 - 2rx} = 17{\cdot}11197$ = DE the diameter of the base of the same segment.

V. Quest. *answered by James Temple, New-York.*

It is well known that a heavy body falling from a state of rest, descends in the first second of time, through a space $s = 16\frac{1}{12}$ feet; and that sound moves

with the uniform velocity $v = 1142$ feet per second. Put x = the height required, then $\frac{x}{v}$ = time of the sounds ascent, and therefore by the question $\frac{x^2}{9v^2}$ = square of the time of the ball's descent. Now since the spaces described by falling bodies are as the squares of the times of falling, we have $s : x :: 1'' : \frac{x}{s} = \frac{x^2}{9v^2}$, and hence $x = \frac{9v^2}{s} = 729791$ feet $= 138{\cdot}218$ English miles the height required.

VI. QUEST. *answered by John Smithis, Philadelphia.*

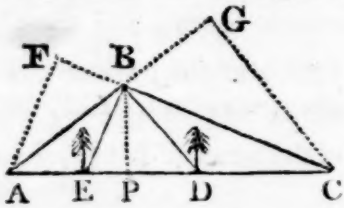
The land will be had at the cheapest rate possible when the sum paid for any given quantity is a minimum. Put $p = 50$ rods = half the given perimiter, and x = the breadth of the rectangle; then will $p - x$ = length, $px - x^2$ = the area, $p + 2x$ = value of the land, and $\frac{p + 2x}{px - x^2}$ = the price of a square rod, = a minimum, the fluxion of which equated with 0 and reduced gives $x^2 + px = \frac{1}{2}p^2$. Whence $x = \frac{1}{2}p \times (\sqrt{3} - 1) = 18{\cdot}30127$, the breadth and $p - x = 31{\cdot}69873$ the length required.

The same answered by the Proposer.

When the quantity purchased for a given sum of money is a maximum the land will be had at the cheapest rate possible. *(Here this ingenious gentleman makes the same substitution as in the preceding solution, and proceeds.)* Therefore $\frac{px - x^2}{p + 2x}$ = quantity purchased for one dollar = a maximum; from the fluxion of which reduced, we have $x^2 + px = \frac{1}{2}p^2$. Hence $x = 18{\cdot}30127$ and $p - x = 31.69873$ the length and breadth required.

VII. *or* PRIZE QUEST. *answered by Robert Adrain, York-Town, Pennsylvania.*

METHOD 1. Let ABC represent the triangular field, AB and BC the given sides, E and D the places of the trees, so that ABD and EBC may be right angles, and ED the given distance of the two trees. Instead of the numbers given in the question it will here be more convenient to take the least integers in the same ratio, viz. 50, 78 and 35, or 5, 7·8 and 3·5; and after obtaining a solution in terms of these numbers, the lengths of the unknown lines will be had by proportion, and consequently the area required will be easily found. This being premised, let AB $= 5 = a$, BC $= 7{\cdot}8 = b$, ED $= 3{\cdot}5 = e$ and BP $= x$; then



$\text{PE} = \frac{\text{BP}^2}{\text{PC}} = \frac{x^2}{\sqrt{b^2 - x^2}}$, and $\text{PD} = \frac{\text{BP}^2}{\text{AP}} = \frac{x^2}{\sqrt{a^2 - x^2}}$. Hence $\frac{x^2}{\sqrt{b^2 - x^2}} + \frac{x^2}{\sqrt{a^2 - x^2}} = e$, or in numbers $\frac{x^2}{\sqrt{60{\cdot}84 - x^2}} + \frac{x^2}{\sqrt{25 - x^2}} = 3{\cdot}5$. Here it is easy to perceive that $x = 3$, and therefore by proportion 5 : 24 :: 3 : 14·4 = the absolute length of the perpendicular BP; consequently the base or third side AC = 53·76, BE and BD the distances of the two trees from the obtuse angle B are 15·6 and 18 respectively, and the area of the field = 387·072 square chains.

COROLLARY.

The sides AB, AP and BP are as 5, 4 and 3 and the sides BC, PC and BP are as 13, 12 and 5.

This discovers the curious artifice used in the composition of this elegant problem.

METHOD 2. Let AB, BC and ED be denoted by a, b and c respectively, also let $b^2 + a^2 = m$, $b^2 - a^2 = n$ and the base AC $= x$; then by the common theorems of geometry we have $CP = \frac{x^2 + n}{2x}$, and $AP = \frac{x^2 - n}{2x}$, likewise $BP^2 = AB^2 - AP^2 = \frac{4a^2x^2 - \overline{x^2 - n}^2}{4x^2}$, $EP = \frac{BP^2}{CP} = \frac{4a^2x^2 - \overline{x^2 - n}^2}{2x \times \overline{x^2 + n}}$, and $DP = \frac{BP^2}{AP} = \frac{4a^2x^2 - \overline{x^2 - n}^2}{2x \times \overline{x^2 - n}}$. But $EP + DP = ED$, or

$$\frac{4a^2x^2 - \overline{x^2 - n}^2}{2x \times \overline{x^2 + n}} + \frac{4a^2x^2 - \overline{x^2 - n}^2}{2x \times \overline{x^2 - n}} = c,$$

and this equation by an easy reduction becomes $x^5 + ex^4 - 2mx^3 + n^2x - en^2 = 0$: which by assuming $a = 50$, $b = 78$ and $c = 35$ gives $x^5 + 35x^4 - 17168x^3 + 12845056x - 449576960 = 0$. Hence by any of the common methods, we find $x = 112$, and therefore by proportion 50 : 24 :: 112 : 53·76 the absolute length of the base AC, from which the other parts of the question are easily found.

The same, answered by the proposer, G. Baron, N. Y.

Let ABC represent the field and D and E the two hemlock trees. Produce AB and CB to G and F, draw CG perpendicular to AG and AF perpendicular to CF. Put AB $= a = 24$, BC $= b = 37{\cdot}44$, DE $= d = 16{\cdot}8$, CG $= x$ and BG $= y$. Then by similar

triangles BC : CG :: AB : AF = $\frac{ax}{b}$, and BC : BG :: AB : BF = $\frac{ay}{b}$. Again by similar triangles AG : GC :: AB : BD = $\frac{ax}{a+y}$ and CF : AF :: CB : BE = $\frac{abx}{b^2+ay}$. And since by the question GBD and CBE are right angles, the angle EBD = supplement of ABC ; of consequence the natural sine of EBD = natural sine of ABC, and therefore by mensuration, the area of the triangle EBD : area of ABC :: BD × BE : BC × BA :: ED : AC = $\frac{\overline{a+y} \times \overline{b^2+ay}}{ax^2} \times d$. Again by mensuration $\frac{AB \times CG}{2} = \frac{ax}{2}$ = area of ABC. Now since the triangle AGC is right-angled at G we have AC = $\sqrt{AG^2 + GC^2} = \sqrt{a^2 + 2ay + y^2 + x^2}$, which equated with the former value of AC gives $\frac{ab^2 + \overline{a^2+b^2} \times y + ay^2}{ax^2} \times d = \sqrt{a^2+2ay+y^2+x^2}$

Further by right angled triangles $x^2 = b^2 - y^2$. From these two equations we have

$$\frac{ab^2 + \overline{a^2+b^2} \times y + ay^2}{ab^2 - ay^2} \times d = \sqrt{a^2 + b^2 + 2ay}.$$

Hence by reduction, and a restoration of the values of the known quantities we obtain an equation from which y is easily found = 19·008 exactly ; therefore $x = \sqrt{b^2 - y^2} = \sqrt{\overline{b+y} \times \overline{b-y}} = 32{\cdot}256$, $\frac{ax}{a+y}$ = BD = 18, $\frac{abx}{b^2+ay}$ = 15·6 = BE, $\sqrt{\overline{a+y}^2 + x^2}$ = 53·76 = AC the base, and $\frac{ax}{2}$ = 387·072 square

chains = 38 acres 2 roods and 33·152 poles the area of the field.

ACKNOWLEDGEMENTS, &c.

The following is an accurate list of the names and places of residence of the ingenious gentlemen who favoured us with true solutions to the questions proposed in No. 1. The figures at the ends of the lines refer to the questions answered by each as numbered in the work.

Robert Adrain, York-Town, Penn.	1, 2, 3, 4, 5, 6 & 7.
George Baron, New-York, -	1, 2, 3, 4, 5, 6 & 7.
John Craggs, near Richmond, Vir.	1, 2, 3, 4, 5, 6 & 7.
Walter Folger, jun. Nantucket, -	6.
Niel Gray, New-York, - -	1, 2, & 7.
William Green, New-York, -	1, 2, & 7.
John Johnson, Essex, Vermont,	1, 2, & 7.
William Lenhart, York-Town, Penn.	3, 4, & 7.
Samuel Moor, New-York, -	1, 2, 3 & 5.
John Smithis, Philadelphia,	1, 2, 3, 4, 5, 6 & 7.
Richard Tagart, New-York,	1, 2, 3, 4, 5, 6 & 7.
James Temple, do. -	1, 2, 3, 4, 5, 6 & 7.
William Thompson, Charleston, S. C.	1, 2, 3, 4, 5, 6 & 7.
Joseph M. White, Danbury, Conn.	7.
Ebenezer R. White, do.	1, 2, 3, 5, & 7.
Diarius Yankee, Bunker's-Hill,	1, 2, 3, 4, 5, 6 & 7.
Titus Youle, Portland, Dist. Maine	1, 2, & 5.
Noah Young, near Lexington, Ken.	1, 2, 3, 4, 5, 6 & 7.

The prize Medal has been determined to Robert Adrain, *York-Town, Pennsylvania.*

N. B. We are sorry that the nature of this work, has compelled us to acknowledge many excellent solutions, which otherwise would have been published at full length. It will always be our particular study to display the talents of our ingenious correspondents, according to their respective merits: those therefore who have been unavoidably neglected in their first attempt, may depend on being remembered in their next, as far as is consistent with the general plan and credit of our work.

As no person who is an editor of this work can receive a prize medal, the merits of the different editors, are not to be considered in awarding the prizes.

ARTICLE I. *continued from page* 15.

§ 21. In the Rule of Proportion, the conditions of the question, and the nature of proportional numbers, must shew that the answer required, is a fourth proportional to three numbers which are given in the question.

Stating the question, is the art of placing the three given numbers in such an order, that the first may be to the second, as the third is to the answer; this is done by the following

General Rule.

Write that number for the third term, which is of the same kind with the answer or number sought. Consider from the nature of the question whether this third term is greater or less than the answer; and if greater write the greater of the other two given numbers for the first term and the less for the second; but if less write the less for the first term and the greater for the second, and the question is stated.*

To find the answer or fourth term,

General Rule.

Case 1. If the first term be a multiple or a part of the second, the third will accordingly be a like multiple, or a like part of the fourth; and the answer will be found by division or by multiplication.

Case 2. If when a part of the first term is added to or subtracted from the first, the sum or the remainder be equal to the second, — a multiple of the second, — or a part of the second; then accordingly a like part of the third term being added to or sub-

* The truth of this rule is demonstrated in Art. 1. § 7, of the Mathematical Correspondent.

tracted from the third, the sum or the remainder will be equal to the fourth term, — a like multiple of the fourth, — or a like part of the fourth. Hence the answer will be known or readily found by case 1. of this rule.

Case 3. In other cases reduce the first and second terms to the same denomination and consider them as pure numbers; then the product of the second and third terms divided by the first, gives the answer in the denomination of the third term.

REMARK.

In the 2d and 3d cases of this rule, the operations will frequently be rendered much shorter, by using like multiples or like parts of the first and second, or of the first and third terms instead of the terms themselves.*

PRACTICAL QUESTIONS.

1. Bought 63 yards of cloth for 143*l.* 6*s.* 6*d.* how much must be paid for 9 yards of the same cloth?

yds.		*yds.*		*l.*	*s.*	*d.*
63	:	9	::	143	6	6

7)143 6 6

20 9 6 Answer.

In this question it is evident that 1 yard is a part of 9 yards, and that the price of one yard is a like part of the price of 9 yards; also that 63 yards are a multiple of 1 yard, and that the price of 63 yards is a like multiple of the price of 1 yard. Hence (by art. 1. § 6 case 3 Math. Corr.) a fourth proportional to the three given numbers is here required: and consequently this and all similar questions belong to

* The first case of this rule depends on Art. 1. § 6 case 1 and 2 M. Corr.
The second case is deduced from Art. 1. sections 8, 9, 10, 11, 12 and 13 Math. Corr.
The third case is founded on Art. 1. § 19 Math. Corr.
The remark is drawn from Art. 1. sections 15, 16, 17 and 18 M. Corr.

the Rule of Proportion. Those who are desirous of acquiring a rational knowledge of this rule, ought, before they attempt the solution of a question, thus to demonstrate to the satisfaction of their own minds, that such a question actually belongs to the rule.

In stating this question, it is easily perceived that the answer required is money; and therefore the given sum of money is here written for the third term. Again, it is evident, that this third term being the price of 63 yards, is greater than the answer or price of 9 yards; consequently 63 the greater of the other two given numbers is written for the first term, and the less 9 for the second: and hence the question is stated according to our general rule.

In finding the answer, since the first term 63 is a multiple of the second 9, of consequence the third term is a like multiple of the fourth; therefore the third term multiplied by 7 gives the answer as above.

2. If 13 bushels of corn cost 12 doll. 75 cents, how much money will purchase 65 bushels at the same rate?

b.		*b.*		*dls. cts.*
13	:	65	::	12 57
				5
				62 85 answer.

3. When the interest of a certain sum of money for 60 days is 85 dolls. 32 cts. what is the interest of the same sum for 75 days?

D.		*D.*		*D. C.*	
60	:	75	::	85 32	*D. C.*
$\frac{1}{4}$ of 60 = 15				21 33	= $\frac{1}{4}$ of 85 32
75				106 65	answer.

Here a part of the first term added to the first is equal to the second, and therefore a like part of the third term added to the third gives the answer.

4. When I can purchase 24 yards of linen for 9*l.* 13*s.* 4*d*, how much ought I to pay for 21 yards of the same?

yds.		*yds.*		*l. s. d.*		
24	:	21	::	9 13 4		*l. s. d.*
$\frac{1}{8}$ of 24 = 3				1 4 2 = $\frac{1}{8}$ of		9 13 4
21				8 9 2		answer.

The question being stated according to the rule, it is easy to see that a part of the first term subtracted from the first, leaves a remainder equal to the second; and consequently a like part of the third term subtracted from the third leaves the answer.

5. I have 45*cwts.* of sugar, the value of which is £64; what quantity of this sugar must I give to discharge a debt of £12?

L.		*L.*		*C. Q. lbs.*	
64	:	12	::	45 0 0	
$\frac{1}{8}$ of 64 = 8				5 2 14 = $\frac{1}{8}$ of	45*cwt.*
72				6)50 2 14	
				8 1 21	answer.

Since a part of the first term, added to the first, is a multiple of the second; a like part of the third term, added to the third, is a like multiple of the answer.

6. Bought 48 tons of logwood for 3960 dollars, how much money would have purchased 7 tons of the same?

T.		*T.*		*Dolls.*	
48	:	7	::	3960	
$\frac{1}{8}$ of 48 = 6				495 = $\frac{1}{8}$ of	3960 dolls.
42				6)3465	
				577 50 *cts.*	answer.

7. How much must be paid for 112*lbs.* of cloves, when 12 *lbs.* may be bought for 9 dolls. 93 cts.

lbs.		*lbs.*		*dolls.*	*cts.*
12	:	112	::	9	93
$\frac{1}{3}$ of 12 = 4				3	31 = $\frac{1}{3}$ of 9 dls. 93 cts.
16				13	24
					7
				92	68 answer.

8. If 9 lbs. of butter cost 12*s.* how many pounds can I have for 2*l.* 16*s.*

s.		*l.*	*s.*		*lbs.*
12	:	2	16	::	9
$\frac{1}{3}$ of 12 = 4					3 = $\frac{1}{3}$ of 9 *lbs.*
8					6
					7
					42 *lbs.* answer.

9. If 15 men can perform a piece of work, in 8 days; how many men must be employed to do the same piece of work, in 6 days?

				Men.
6	:	8	::	15
$\frac{1}{3}$ of 6 = 2				5 = $\frac{1}{3}$ of 15 *men.*
8				20 answer.

As this is *apparently* of a nature, different from that of the eight preceding questions, and as the authors of the books of arithmetic used in the American schools appear to have been entirely ignorant of the *real* nature of such questions, I am particularly anxious that the inquisitive reader may form a clear understanding of the following remark.

It is evident from the question that 15 men can do six times this work in 8 × 6 or 48 days; and that the number of men required, must do eight times the same work in 6 × 8, or 48 days. Hence then

the question amounts to this : *If* 15 *men be capable of performing a piece of work* 6 *times in a certain number of days, how many men can do that piece of work* 8 *times in the same number of days ?* It is now manifest that this and all other similar questions are theoretically of the same nature with those already answered in this article ; and may therefore be stated and resolved according to our general rules. After the true theory of the resolutions of such questions, has thus been established in the mind of the learner ; he will readily perceive that the same questions, when taken in a literal sense, may always be truly stated by the general rule. For in the present question, the number required is men ; and therefore the given number 15 men is here written for the third term. Again, since a less number of men could do the work in 8 days, than in 6 ; this third term is less than the answer, and consequently the less of the other two given numbers 6, is written for the first term, and the greater 8, for the second : hence the question is rightly stated.

What is here remarked, is equally applicable to the two next succeeding questions, and to others of a similar import.

10 By travelling 16 hours a day I performed a certain journey in 28 days ; how long would I have been on the same journey had I travelled but 7 hours a day ?

Days.

7 : 16 :: 28

1 : 16 :: 4 (by art. 1. § 18. Math. Corr.)

4

64 *days* answer.

11 How many pounds of bacon at 8 cents a pound must be given for 342 lbs. of English cheese at 28 cents per pound ?

				lbs.
8	:	28	::	342
4	:	28	::	171
				7
				1197 *lbs.* answer.

12 How much must be paid for 8*cwts.* 3 *qrs.* 25*lbs.* of pig-iron, at the rate of 28 dollars per ton?

C.		*C. qr. lb.*		*dolls.*
20	:	8 3 25	::	28
112		4		
2240		35		
		28		
		305		
		70		
320	:	1005	::	4
80	:	1005	::	1

8,0)100,5

12 *dolls.* 5*d.* 6*c.* 2*m.* 5*t.m.* Or, 12 dollars, 5 dimes, 6 cents, 2 mills, and 5 of the tenth parts of a mill; the answer in federal money.

13. How much must I pay for 219 days' board, at the rate of 200*l.* 15*s.* 10*d.* a year?

D.		*D.*		*l.*	*s.*	*d.*
365	:	219	::	200	15	10
73	:	219	::	40	3	2
						3
				£120	9	6 ans.

14. If my board for 219 days cost £120 9 6, how long can I be boarded for £200 15 10, at the same rate?

l.	*s.*	*d.*		*l.*	*s.*	*d.*		*D.*
120	9	6	:	200	15	10	::	219
40	3	2	:	200	15	10	::	73
								5
								365 days, answer.

15 The carriage of 52 *cwts.* 2 *qrs.* 18 *lbs.* for 120 miles, cost me £15 12; how far can I have 43 *cwts.* 3 *qrs.* 15 *lbs.* carried for the same money?

c. qr. lb.		*c. qr. lb.*		*M.*
43 3 15	:	52 2 18	::	120
8 3 3	:	52 2 18	::	24
				6
				144 miles, answer.

16. If 1 *cwt.* of sugar cost £6 17, what must I pay for 13 *cwts.* 3 *qrs.* 14 *lbs.* of the same?

c.		*c. qr. lb.*		*l. s.*
1	:	13 3 14	::	6 17
		4		20
		55		137
		28		
		454		
		110		
112	:	1554		
16	:	222		
8	:	111	::	137 shillings.
				111
				137
				1507
				8)15207
				2,0)190,0 $10\frac{1}{2}$ *d.*
				£95 0 $10\frac{1}{2}$ answer.

INVENTION OF GENERAL RULES.

17. How much federal money is equal to any given number of crowns: suppose 498?

				*dolls.**
	10	:	11	:: 498 *cts.*
$\frac{1}{10}$ of 10 =	1			49 80 = $\frac{1}{10}$ of 498.
	11			$547 80

* It is evident from the nature of this question, that the answer must be a fourth proportional to 10, 11 and 498 dolls. and hence the numbers are stated in this order. For similar instances, see the following questions.

General Rule. The given number of crowns, considered as dollars, and increased by $\frac{1}{10}$ of itself, gives the federal money required.

18 It is required to find how much New-England currency is equal to any given sum of sterling money: suppose £678 8 6?

Here since 9 sterling sixpences = 1 dollar = 12 sixpences N. England currency;

			l. *s.* *d.*	
9	:	12 ::	678 8 6	
$\frac{1}{3}$ of 9 = 3			226 2 10 = $\frac{1}{3}$ of £678 8 6.	
12			904 11 4 N. Eng. curr. ans.	

General Rule. Any sum of sterling money, increased by $\frac{1}{3}$ of itself, gives the New-England currency required.

Corollary 1. This rule will also serve to reduce New-England currency to that of New-York.

Corollary 2. Hence $\frac{1}{4}$ of any sum of New-England currency, subtracted from that sum, leaves the value of the same sum, in sterling money. And exactly in the same manner, may any sum of New-York currency be reduced to its equivalent in the currency of New-England. Thus,

l.	*s.*	*d.*		*l.*	*s.*	*d.*	
4)904	11	4	N. Eng.	4)5616	18	8	N. York.
226	2	10		1404	4	8	
678	8	6	sterling.	4212	14	0	N. Eng.

N. B. The method of reducing sterling money to New-York currency, and the reverse; has already been shewn in page 21 and 22, Math. Corr. Questions concerning the reductions of other currencies, in the United States, &c. will hereafter be proposed

in this work, and answered by some of our contributors.

19 What is the tare on a certain number of boxes of sugar, weighing in all 55 *cwts.* 2 *qrs.* 19 *lbs.* at 15 per cent ?

					c.	*qr.*	*lb.*
100	:	15	::	55	2	19	
20	:	15	::	11	0	15	
$\frac{1}{4}$ of 20 = 5				2	3	$3\frac{3}{4}$	
15				8	1	$11\frac{1}{4}$	answer.

General Rule. Divide the gross weight by 5 ; and $\frac{1}{4}$ of the quotient, subtracted from the quotient leaves the tare at 15 per cent.

Otherwise,

				c.	*qr.*	*lb.*	
100	:	15	::	55	2	19	
$\frac{1}{2}$ of 100 = 50				27	3	9	8 *oz.*
150				10)83	2	0	8
				8	1	11	4 answer.

General Rule. The gross weight increased by $\frac{1}{2}$ of itself gives 10 times the tare, at 15 per cent.

20 I import a certain article from England which I sell at the rate of 75 per cent. advance on the invoice ; how much in New-York currency ought I to receive for a quantity of this article, invoiced at £758 15 6 sterling ?

Here it is to be observed that £9 sterling is equal to £16 New-York currency ; and that 16 at 75 per cent. advance amounts to 28.

				l.	*s.*	*d.*	
Hence 9	:	28	::	758	15	6	
27	:	28	::	2276	6	6	
$\frac{1}{27}$ of 27 or $\frac{1}{9}$ of 9 = 1				84	6	2	
28				2360	12	8	answ.

General Rule. Multiply the invoice sum of sterling money by 3, and this product increased by $\frac{1}{9}$ of the same invoice sum, gives the New-York currency required.

21. The invoice of a certain quantity of goods amounted to £948 18 sterling, and I have sold the same goods at the rate of 50 per cent. advance on the invoice; what is their amount in New-York money?

				l.	*s.*
9	:	24	::	948	18
18	:	24	::	1897	16
				632	12
				£2530	8

General Rule. Multiply the given invoice in sterling money by 2, and this product increased by $\frac{1}{3}$ of itself, is the New-York currency required.

Having now pursued my subject as far as the bounds of this work will permit; I conclude by requesting the diligent reader, to make himself perfectly master of the elementary principles contained in the first 20 sections of this article. For when these are well understood, their practical application will be easily acquired: and here the learner will find his labours amply rewarded, in the discovery of thousands of simple and elegant rules, of the greatest utility in the busy affairs of life.

I intend in some future number to give a paper on the elements of Fractions and Decimals, in which I shall have an opportunity of exhibiting a more extended application of the elementary principles of the Rule of proportion.

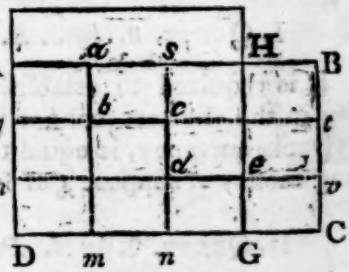
G. BARON.

New-York, April 20, 1804.

ARTICLE VI.

A PROBLEM, *by James Temple, New-York.*

To determine in what cases it is possible to cut a given rectangle ABCD into two parts, which being joined together, in a certain manner, shall form a square, equal to that rectangle.

By the 14th prop. of the 2d book of Euclid, make the square EFGD = ABCD the given rectangle; then the rectangle EH is evidently equal to the rectangle HC: and by the 14th prop. of the 6th book of Euclid, AH (or DG) : HB (or GC) :: GH (or AD) : FH (or AE). But from the nature

of proportional quantities, if DG be a multiple of GC; AD will be a like multiple of AE: and whenever this is the case, the problem may be done in the following manner. Divide DG into a certain number of equal parts, in the points *m*, *n*, &c. so that each part may be equal to GC. Divide AD into the same number of equal parts in *p*, *q*, &c. and each of these parts will be equal to AE. Through *m*, *n*, &c. draw *ma*, *ns*, &c. parallel to AD; and through *p*, *q*, &c. draw *pv*, *qt*, &c. parallel to DG. Let the given rectangle ABCD be cut along the indented line *abcde*G and the problem is done. For by construction and hypothesis, the figure EF*edcba*AE is similar, and in every respect, equal to the figure *a*BCG*edcba*; if therefore the latter be applied to the former, so that *a* may coincide with E, *a*B with EF, and BC with Fe; the indented line *abcde*GC will then coincide with the indented line A*abcde*: and consequently the given rectangle ABCD has been cut into two parts which being

thus joined together, form the square EFGD = the same rectangle as was required to be done. Hence it is easy to perceive that when DG is not a multiple of GC the problem is impossible.

ARTICLE VII.

NEW QUESTIONS to be answered in the next number.

I. QUEST. 8. *by J. B. Brewer, New-York.*

It is required to establish a general rule, for finding in the shortest manner possible, how much Pennsylvania currency, is equal to any given sum of sterling money : suppose £876 19.

II. QUEST. 9. *by A. Picket, New-York.*

I demand a general rule, for finding in the easiest manner possible how much sterling money is equal to any given sum in Pennsylvania currency : suppose £436 15 10.

III. QUEST. 10. *by Niel Gray, New-York.*

The commanding officer of a certain post, being sick, was informed by a physician, that if five handfuls of a certain herb, could be immediately procured, the disease might be removed. The officer ordered five of his soldiers A, B, C, D and E, to go instantly in quest of the herb ; and, as an inducement to return as quickly as possible he promised to divide $20 amongst them in a reciprocal proportion to the number of minutes in which each should present him with a handful of the herb. A returned in 5, B in 7, C in 9, D in 11 and E in 12 minutes : what share of the $20 has each soldier to receive?

IV. QUEST. 11. *by Samuel Moor, New-York.*

A certain tract of land, exactly square, is enclosed by a three railed fence; the length of each rail is 15 feet, and the number of rails in the fence is equal to the number of acres enclosed. I demand the area of this tract in acres and the length of its side in feet.

V. QUEST. 12. *by Diarius Yankee, Bunker's-Hill.*

In the Highlands of Scotland, the head of any old ram is but just capable of resisting a blow, the intensity of which is equal to the momentum acquired by a body of 30 pounds weight, in falling freely thro' the height of 20 feet. Now a duel is to be fought between two such rams, the one weighing 54, and the other 57 pounds; and it is required to find the respective velocities with which they must meet, so that at the first blow, each may beat out the brains of the other: both remaining perfectly at rest after that blow.

VI. QUEST. 13. *by John Smithis, Philadelphia.*

A ladder is placed perpendicular to the plane of the horizon, and in coincidence with the plane of an upright wall. If the base of the ladder be drawn along the horizontal plane, in a direction perpendicular to the plane of the wall, the top of the ladder sliding downwards, against the wall; it is required to find the equation of the curve which is the locus of a point taken any where on the ladder.

VII. QUEST. 14. *by G. Baron, New-York.*

Given $\frac{f\dot{x}}{x} + \frac{r\dot{y}}{y} = \frac{x^m\dot{x}}{ay^n}$ to find the general relation of the fluents.

VIII. QUEST. 15. *by R. Tagart, New-York.*

The difference of latitude of two ports A and B, both north of the equator, is 1800 minutes, the meridional distance of A from B is 652·5, and the meridional distance of B from A is 413 minutes; required the latitudes and difference of longitude of the two ports.

IX. QUEST. 16. *by W. Thompson, Charleston, S. C.*

Given r = the radius of a circle described by a horse in turning a machine, and l = the length of the traces, or the distance of the horse from that end of the beam to which he is yoked; to find x = the length of the same beam, so that the horse may turn the machine with the greatest ease possible.

X. QUEST. 17. *by G. Baron, New-York.*

Given r = the radius, v = the versed sine, and s = the sine of a circular arc; to discover a more simple theorem than has hitherto been published, for determining t = the tangent of half that arc.

XI. PRIZE QUEST. 18. *by James Temple, New-York.*

In lat. 40 deg. 40 min. N. I observed a transit of the star Arcturus, over the eastward vertical semi-circle, which coincided with the plane of an upright wall. I afterwards observed a transit of the star Antares over the same vertical semi-circle: when I found by my watch, which kept good solar time, that 15 min. 45 sec. had elapsed between these two transits. The right ascension of Arcturus was then 14 h. 6 m. 44 sec. and declination 20 deg. 12 min. 19 sec. N. the right ascension of Antares 16h. 17 m. 26 sec. and declination 25 deg. 59 min. 01 sec. S. Required the azimuth and declination of the wall.

ARTICLE VIII.

ANSWERS to the Questions proposed in Article VII.

I. QUEST. 8. *answered by the proposer J. B. Brewer New-York.*

BECAUSE 9 sterling sixpences are equal to 15 sixpences, in Pennsylvania currency; we have,

9 : 15, or 3 : 5 :: 876*l.* 19*s.* : sum required.

5

3)4384 15

1461*l.* 11*s.* 8*d.* answer.

General Rule. Multiply any sum of sterling money by 5; and one third of the product, gives the Pennsylvania currency required.

OTHERWISE.

		l.	*s.*		
9	: 15 ::	876	19	: sum sought.	
$\frac{1}{3}$ of 9 = 3		292	6	4*d* = $\frac{1}{3}$ of 876*l.* 19*s.*	
ditto. = 3		292	6	4 = ditto	
15		1461	11	8 answer.	

General Rule. Any sum of sterling money increased by $\frac{2}{3}$ of itself, gives the Pennsylvania currency required.

II. QUEST. 9. *answered by Alexander Walsh, New-York.*

l. *s.* *d.*

15 : 9, or 5 : 3 :: 436 15 10 : sum required.

3

5)1310 07 6

262 01 6 answer; and hence the general rule is evident.

OTHERWISE.

l. *s.* *d.*

5 : 3 :: 436 15 10 : sum sought.

$\frac{1}{5}$ of 5 = 1 87 7 2 = $\frac{1}{5}$ of the given sum.

6 2)524 03 0

262 01 6 answer.

General Rule. Any sum of Pennsylvania currency, increased by $\frac{1}{5}$ of itself, gives double the sterling money required.

III. QUEST. 10. *answered by Gad Dumbolton, New-York.*

Since it is known, that, the reciprocals of numbers are reciprocally, as the numbers themselves; the required shares must be as $\frac{1}{5}$, $\frac{1}{7}$, $\frac{1}{9}$, $\frac{1}{11}$, and $\frac{1}{12}$; or as $\frac{1}{5} \times 13860$, $\frac{1}{7} \times 13860$, $\frac{1}{9} \times 13860$, $\frac{1}{11} \times 13860$, and $\frac{1}{12} \times 13860$; where 13860 is the least common multiple of 5, 7, 9, 11 and 12; and therefore 2772, 1980, 1540, 1260, and 1155, are the least integers, which are directly as the shares sought. But 2772 + 1980 + 1540 + 1260 + 1155 = 8707; and hence by proportion,

$$\$707 : \left\{\begin{matrix}2772\\1980\\1540\\1260\\1155\end{matrix}\right\} :: \$20 : \left\{\begin{matrix}\$\ cts.\\6\ 36\ \frac{6148}{8707} = \text{A's share}\\4\ 54\ \frac{7022}{8707} = \text{B's}\\3\ 53\ \frac{6429}{8707} = \text{C's}\\2\ 89\ \frac{3677}{8707} = \text{D's}\\2\ 65\ \frac{2645}{8707} = \text{E's}\end{matrix}\right.$$

Proof 20 09

N. B. This question is of a similar nature, with quest. 9, in page 96 of the Columbian Accountant. The author, Mr. Shepherd, has answered that question, on erroneous principles. His answers are $\$ 20{\cdot}83\frac{1}{3}$, $\$ 16{\cdot}66\frac{2}{3}$, and $\$ 12{\cdot}50$; the true answers are $21{\cdot}27\frac{31}{47}$, $15{\cdot}95\frac{25}{47}$ and $12{\cdot}76\frac{28}{47}$ *dolls.* This is but a small specimen of what we could give, of the numerary talents of Shepherd.

Editors.

The same answered, by the proposer, Niel Gray, New-York.

Put $\frac{1}{5}x$ = A's share; then per q. $7 : 5 :: \frac{1}{5}x : \frac{1}{7}x$ = B's, $9 : 7 :: \frac{1}{7}x : \frac{1}{9}x$ = C's, $11 : 9 :: \frac{1}{9}x : \frac{1}{11}x$ = D's, and $12 : 11 :: \frac{1}{11}x : \frac{1}{12}x$ = E's. Also per q. $(\frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{12}) \times x = \$20$. Whence $x = \$\frac{277200}{8707}$,

	dol.	*cts.*	
$\frac{1}{5}x$ =	6	$36\ \frac{6148}{8707}$	= A's share,
$\frac{1}{7}x$ =	4	$54\ \frac{7022}{8707}$	= B's,
$\frac{1}{9}x$ =	3	$53\ \frac{6429}{8707}$	= C's,
$\frac{1}{11}x$ =	2	$89\ \frac{3677}{8707}$	= D's,
and $\frac{1}{12}x$ =	2	$65\ \frac{2645}{8707}$	= E's.

IV. Quest. 11. *answered by the Rev. Thos. P Irving, Newbern, North Carolina.*

Put $x = \frac{1}{12}$ of the number of rails in the fence; then per quest. $15x$ = the length of the side, in

feet; and $12x$ = the area of the field in acres, = $\frac{1}{160} \times \left(\frac{15x}{66}\right)^2$, by mensuration. Whence $x = \frac{12 \times 968}{5}$, $15x = 34848$ feet, the length of the side, and $12x = 27878{\cdot}4$ acres, the area of the field.

The same answered by John Capp, Harrisburg, Pennsylvania.

Suppose the tract of land to be inclosed by one rail; then per quest. $\frac{15}{4 \times 3}$, or $\frac{5}{4}$ feet the length of its side; and by mensuration $\frac{25}{16}$ = the area in square feet. Now, since 43560 square feet make an acre, we have $\frac{25}{16} : 43560 :: 1$ rail : $27878{\cdot}4$ the number of rails in the fence, or the number of acres in the tract; and $\frac{27878{\cdot}4 \times 15}{4 \times 3}$, $= 34848$ the length of the side in feet.

Scholium by the Editors.

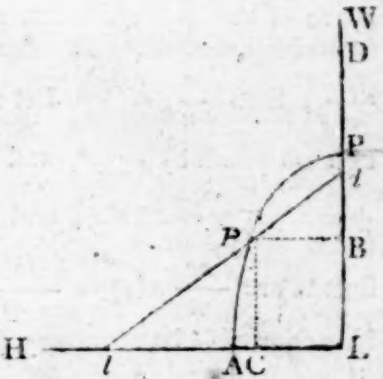
We think proper, here, to demonstrate the curious proportional theorem, used in this solution, by Mr. Capp. Let $l = 15$ feet, the length of a rail, $n = 3$, $f = 43560$, the square feet in an acre, and x = the number of rails in the fence, or the number of acres in the tract; then is $4\sqrt{fx} = \frac{lx}{n}$: hence $\frac{x \times l^2}{16n^2} = f$, or $\frac{l^2}{16n^2} : f :: 1 : x$. *Q. E. D.*

V. QUEST. 12. *answered, by the proposer, Diarius Yankee, Bunker's Hill.*

By the known laws of falling bodies, $\sqrt{16\frac{1}{12}} : \sqrt{20} :: 32\frac{1}{6} : \frac{2}{3}\sqrt{2895}$, the velocity acquired by a heavy body in falling freely from the height of 20 feet. Again from the conditions of the question, and the laws of motion, we have $54 \times 2 : 30 :: \frac{2}{3}\sqrt{2895} : \frac{5}{27}\sqrt{2895} = 9{\cdot}9639$ and $57 \times 2 : 30 :: \frac{2}{3}\sqrt{2895} :: \frac{10}{57}\sqrt{2895} = 9{\cdot}4395$. Hence then if the heavier ram, running at the rate of 9·4395 feet per second, meet the other, running at the rate of 9·9639 feet per second; they will strike each other, with equal momentums, and their heads are but just capable of resisting that blow: if, therefore, these velocities, be increased by two indefinitely small quantities, which, are respectively as the velocities themselves, each ram will perform the conditions of the question.

VI. QUEST. 13. *answered by Walter Folger, Jun. Nantucket.*

Let HL represent the horizontal plane, WL the upright wall, and LD the ladder, on which take any point P. Put $PL = a$, and $DP = c$. Let now the ladder be drawn into the position dl; then the point P, will in moving from P to p, describe the curve Pp. Draw pB perpendicular to PL, and pC perpendicular to HL: put $PB = x$, and $pB = y$; then is BL or $pC = a - x$.



Now by similar triangles $pd = PL : pd = DP : : lC : pB$; or $a : c : : \sqrt{a^2 - \overline{a - x}^2} : y$; or $a^2 : c^2 : : 2ax - x^2 : y^2$; hence $a^2y^2 = c^2 \times \overline{2ax - x^2}$: and therefore by the conic-sections, the curve PpA is an ellipsis, whose axes are PL and PD, or PL and AL.

COROLLARY.

When PL = PD, the curve PpA is a circle, whose radius is equal to half the length of the ladder. For when PL = PD, or when $a^2 = c^2$; $a^2y^2 = c^2 \times \overline{2ax - x^2}$, becomes $y^2 = 2ax - x^2$, the known equation of a circle, whose radius is a.

VII. QUEST. 14. *answered by the proposer, G. Baron, New York.*

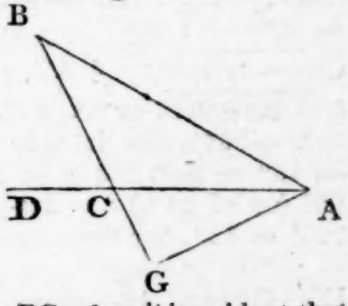
The given equation, $\frac{p\dot{x}}{x} + \frac{r\dot{y}}{y} = \frac{x^m\dot{x}}{ay^n}$, is the second example in art. 262, Simpson's Fluxions; where that learned author has found the relation of the fluents, in a particular case, when $n = r$. Some years ago I discovered, that, the general relation of the same fluents, might be found, by the following method. Assume $r : n : : p : d = \frac{np}{r}$; then $\frac{r}{n} = \frac{p}{d}$. Let now the given equation be multiplied by ax^dy^n, and it becomes $apy^nx^{d-1}\dot{x} + arx^dy^{n-1}\dot{y} = x^{m+d}\dot{x}$; and since $\frac{ap}{d} = \frac{ar}{n}$, the fluents are $\frac{ar}{n}x^dy^n = \frac{1}{m+d+1} \cdot x^{m+d+1}$; which if no correction be necessary, may be reduced to $y^n = \frac{n}{a} \cdot \frac{1}{np + mr + r} \cdot x^{m+1}$.

COROLLARY.

The first example, in the article above mentioned, is $\frac{\dot{x}}{x} + \frac{\dot{y}}{y} = \frac{x^m \dot{x}}{ay^n}$: here Mr. Simpson is evidently mistaken, in supposing, that, the relation of the fluents, can only be found in this way, when $n = 1$. For since $p = r = 1$; we have $y^n = \frac{n}{a} \cdot \frac{1}{n+m+1} \cdot x^{m+1}$: the general relation of the fluents, when no correction is necessary.

VIII. QUEST. 15. *answered by Robert Adrain, York-Town, Pennsylvania.*

The meridional distances, being as the sines of the co-latitudes, we may construct the problem as follows. Draw two straight lines BA and AD, making the angle BAD equal to 30°, the given difference of the co-latitudes. Make BA and AC equal to the given mer. distances. Join BC; then it is evident that ABC and BCD = 35° and 65°, are the co-latitudes; or the latitudes required are 55° and 25° hence the difference of longitude is easily found = 12°.



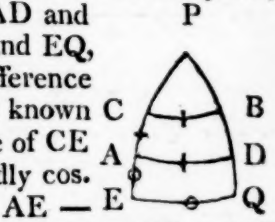
Or thus. The meridional distances, being inversely as the secants of the latitudes, therefore make BA and AC equal to the given meridional distances, containing the angle BAC = 30°, the given difference of latitude. Join BC, on which produced, (if necessary,) let fall the perpendicular AG; then CAG and BAG = 25° and 55° are the latitudes sought and consequently the difference of longitude is 12°.

COROLLARY.

If the places lie on different sides of the equator, the perpendicular AG, will fall within the triangle BAC.

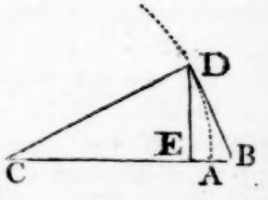
The same answered by James Temple, New-York.

Let P be the north pole, A and B the two ports, PE and PQ their meridians, AD and BC their parallels of latitude, and EQ, an arc of the equator, their difference of longitude. Then by a well known theorem, cosine of AE : cosine of CE : : AD : BC; and compoundedly cos. of AE + cos. of CE : cos. of AE — cos. of CE : : AD + BC = 1065·5 : AD — BC = 239·5 : : (by Trigonometry) cotangent of $\frac{1}{2}$ · CE — AE (or of $\frac{1}{2}$ AC, or of 15°,) : tangent of $\frac{1}{2}$ · CE + AE, or of 40°. Hence CE = 40° + 15° = 55°, the latitude of B; and AE = 40° — 15° = 25°, the latitude of A. Lastly cosine of CE (or of 55°,) : radius : : BC = 413 : EQ = 720′ = 12°, the difference of longitude required.



IX. QUEST. 16. *answered by the proposer, W. Thompson, Charleston, South Carolina.*

Let CD = CA = r = the given radius of the circle described by the horse, CB = x = the length of the beam, and DB = l = the length of the traces, which may also denote the force exerted by the horse in the direction DB. Draw DE perpendicular to CB; then by a property of the triangle, (supposing CD greater than DB,) we have CE — EB =

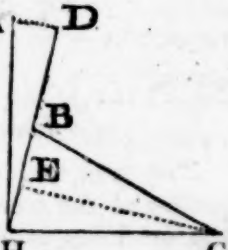


$\frac{r^2 - l^2}{x}$, and CE $= \frac{r^2 - l^2 + x^2}{2x}$; consequently DE $= \frac{1}{2x} \times \sqrt{4r^2x^2 - \overline{r^2 - l^2 + x^2}^2}$. Now by a well known mechanical property, and by the conditions of the question, CB $\times$ DE $= \frac{1}{2}\sqrt{4r^2x^2 - \overline{r^2 - l^2 + x^2}^2}$ = a maximum; the fluxion of which equated with 0 gives $x = \sqrt{r^2 + l^2}$: and hence CDB is a right angle.

OTHERWISE.

Let the figure represent as in the above solution; then by mechanics, the power of the horse, to turn the machine, is as CB $\times$ DE. Hence this power is the greatest possible, when the area of the triangle CBD is a maximum. Let n = the natural sine of the angle CDB; then by mensuration, the area of the triangle CDB = CD $\times$ DB $\times \frac{1}{2} n$. But CD $\times$ DB $\times \frac{1}{2}$ is a constant quantity, and therefore the area of the triangle is a maximum, when n is a maximum; which is when CBD is a right angle, as before.

The same answered by Robert Adrain, York-Town, Pennsylvania.

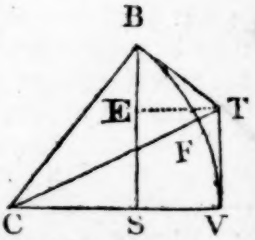


Let CH, HB, BC, be the radius, traces, and beam. Draw HA at right angles to CH, and equal to it; and on HBD, let fall the perpendiculars AD and CE. Let AH represent the absolute force of the horse, in the direction AH; then by mechanics HD is his action on the traces HB; and again

the action of the traces, on the beam at B is as $HD \times CE = \overline{HD}^2 = \overline{CE}^2$, which evidently increases as DH approaches AH; and is a maximum when they coincide; that is when BHC is a right angle: therefore $x = \sqrt{r^2 + l^2}$.

Corollary. The effective force of the horse, is the same whatever the length of the traces may be, when their position is given or constant: for that force is as the square of the sine, of the angle CHB.

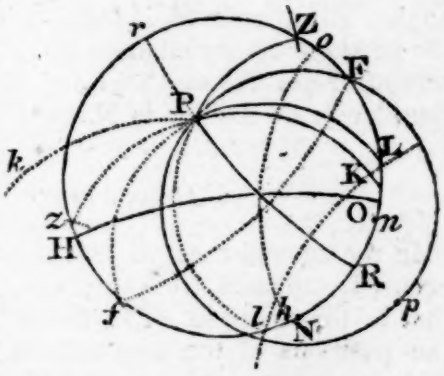
X. QUEST. 17. *answered by R. Tagart, New-York.*



Let BV be the arc, CB = CV = r the given radius, BS = s the given sine, and VS = v the given versed sine. Let the straight line CF bisect the arc BV in F; draw BT perp. to CB intersecting CF, produced in T; then is BT, the tangent of half the arc BV. Join TV which, as is well known, will be perpendicular to CV, or parallel to the sine BS: through the point T, draw TE parallel to CV intersecting BS in E. Then in the similar triangles CBS and BTE, BS, : BC : : TE = VS : BT; or $s : r :: v : t$. Hence $t = \frac{rv}{s}$ or log. t = log. r + log. v — log. s, the theorem required.

XI. PRIZE QUEST. 18. *answered by the proposer, James Temple, New-York.*

The annexed figure is the stereographic projection of the concave sphere, on the plane of PF*ft*, the meridian of Arcturus; where P represents the north pole, and *t* the south pole. Having described



Ff, the parallel of declination of Arcturus, 20° 12′ 19″ N. and Ll the parallel of declination of Antares, 25° 59′ 01″ S; make the angle FPK (2*h*. 10*m*. 42 *sec*.) = 32° 40′ 30″, = the nearest distance of the stars, in right ascension, by drawing PK, the meridian of Antares, forward from PF; then F is the point of the first transit, and K is the cotemporary position of Antares. But by the diurnal rotation, Antares is apparently translated from K, along KL, to the azimuth at L, in 15 *min*. 45 *sec*. solar time, or 15*m*, 47 *sec*. 35*thds*. siderial time; which is equal to 3° 56′ 54″ = the angle KPL, made by drawing the meridian PL, backward, from PK; then L is the point of the last transit. Through the points F and L, draw the azimuth ZFLN*z*; and round P, at the distance PZ, the colatitude = 49° 20′ N. draw the parallel of latitude of the place of observation, to cut the azimuth in Z or *z*, the zenith; through which and P, draw PZ, or P*z*, the meridian of the place. Then is ZFL, or *z*FL, the vertical required; or the angle PZF, or P*z*F, is the azimuth from the north, on the same side of the meridian PZ, or P*z*; and since PZ = P*z*, the angle P*z*F = PZ*f*, the supplement of PZF. But in this question, it is impossible to be the acute azimuth P*z*F, and can only be the obtuse azimuth PZF; for by drawing *ho*, the horizon of *z*, and HO the horizon of Z, we see both the transits F and L fall below *ho*, and above HO. Therefore the

angle PZF, measured, is the required azimuth from the north, or its supplement PZ*f*, measured, is the azimuth from the south = 41° 23′ 11″ E. Hence the plane's declination is N. 48° 36′ 49″ E.

CALCULATION.

In the several cases of this problem, which may occur, it appears from the circles of the sphere, that in the triangle FPL, formed by the pole, and the positions of the two transits; the polar angle FPL is always equal to the difference between the siderial elapsed time, and the distance of the stars in right ascension, counted in *consequentia* from that star which makes the first transit; or else the supplement of that difference, to 24 hours, when it exceeds 12 hours. Also it is evident in this question, that the angle FPL, = FPK — KPL = 1*h*. 54*m*. 54*s*. 25*thds*. = 28° 43′ 36″, and the including sides PF and PL, the stars' polar distances are given. Perpendicular to the base or azimuth FL, draw the meridian *r*PR, which falls nearest the greater side PL, when the greater declination is toward the depressed pole *p*. Then, by cor. 2. prop. 30, book 3, Emerson's Trigonometry, sine of (PL + PF) : sine of (PL ~ PF) : : cotangent of half the vertical angle FPL (or of ½ · $\overline{\text{FPR} \mp \text{LPR}}$) : tangent of ½ · $\overline{\text{FPR} \mp \text{LPR}}$, according as the perpendicular falls within or without the triangle PFL, or according as the acute arc of the 4th term is less or greater than that of the 3d. But sine $\overline{\text{PL} + \text{PF}}$: sine $\overline{\text{PL} \sim \text{PF}}$. : sine $\left\{\begin{matrix}\text{sum}\\\text{diff.}\end{matrix}\right\}$ of the declinations : sine of their $\left\{\begin{matrix}\text{difference when the same way}\\\text{sum when contrary ways}\end{matrix}\right\}$: : cotangent ½ · FPL (or of ½ · $\overline{\text{FPR} \mp \text{LPR}}$) :

tan. $\frac{1}{2}$. $\overline{\text{FPR} \mp \text{LPR}}$; and consequently FPR and LPR are the sum and difference of the arcs in the third and fourth terms. Therefore in this question, sine 5° 46′ 42″ : sine 46° 11′ 20″ : : cotangent 14° 21′ 48″ : tangent 87° 57′ 14″; whence the angle LPR = 73° 35′ 26″. Now in the right angled triangle LPR, the angle LPR and the side PL are given, to find the perpendicular PR = 149° 54′ 8″. And lastly in the right angled triangle ZPR, the sides PR and PZ, are given to find the angle PZR, the azimuth N. 138° 36′ 49″ E. or PZ*r* = the azimuth from the South 41° 23′ 11″ E. and consequently the plane's declination is N. 48° 36′ 49″ E.

Or in the triangle PFL the sides PF and PL and the angle FPL are given, to find the angle PLF; then in the triangle PZL the sides PZ and PL and the angle PLF are given, to find the angle Z, the azimuth N. 138° 36′ 49″ E. or S. 41° 23′ 11″ E.

Corollary 1. If the two transits be made, at the same instant, the angle FPL, is equal to the least distance of the two stars, in right ascension.

Cor. 2. If the polar distances of the two stars be equal, and the two transits be made on the same quadrant ZO, the azimuth cannot be found by this problem: for when PF = PL, the points F and L coincide, and the triangle FPL vanishes.

Cor. 3. If a star in rotation, be observed just to touch an azimuth circle, the azimuth may be easily found from that observation. For if a star *r*, touch an azimuth circle FZ*z* at *r*, the triangle P*r*Z is right angled at *r*, and hence the sine of PZ : rad. : : sine of P*r* : sine PZ*r*; or cosine lat. : rad. : :

cosine star's declination : sine of the star's greatest azimuth ; or rad. : secant lat. : : cosine star's declination : sine of the star's greatest azimuth.

Cor. 4. Since any star, whose polar distance, is greater than the complement of the latitude, will in its rotation, touch some azimuth circle, when the azimuth of that star becomes the greatest possible ; and since it is evident, that, the star's magnetic azimuth may then be accurately observed, we are, therefore, in cor. 3. furnished with a new, easy, and correct method of finding the variation of the compass, at land.

N. B. Six other curious corollaries are here omitted for want of room. We hope, that, Land-Surveyors will pay particular attention to cor. 3 *and* 4.

EDITORS.

ACKOWLEDGEMENTS, &c.

The following is an accurate list of the names and places of residence of the ingenious gentlemen who favoured us with true solutions to the questions proposed in No. II. The figures at the ends of the lines, refer to the questions answered by each, as numbered in the work.

R. Adrain, York-Town, Penn.	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11.
G. Baron, New-York, -	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11.
J. B. Brewer, New-York,	1, 2.
J. Capp, Harrisburg, Penn.	3. 4.
J. Craggs, near Richmond, Vir.	1, 2, 3, 4, 5, 6, 8, 9, 10, 11.
Gad Dumbolton, New-York,	3,
W. Folger, jun. Nantucket,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10.
Niel Gray, New-York.	1, 2, 3, 4, 8.
William Green, New York,	4, 11.
Rev T P. Irving, Newbern, N.C.	1, 2, 3, 4, 5,
Rusticus, New-Brunswick, N. J.	10, 11.
J. Smith, Philadelphia,	1, 2, 3, 4, 5, 6, 8, 9, 10, 11.
R. Tagart, New-York,	4, 8, 10.
J. Temple, New-York,	1, 2, 3, 4, 5, 6, 8, 9, 10, 11.

W. Thompson, Charleston, S C.	1, 2, 3, 4, 5, 6,	8, 9, 10, 11.
Alex Walsh, New-York,	1, 2, 3, 4, 5.	
Darius Yankee, Bunker's Hill,	1, 2, 3, 4, 5, 6,	8, 9, 10, 11.
Titus Youle, Portland, Maine,	1, 2, 4,	
N. Young, near Lexington, Ken.	1, 2, 3, 4, 5,	10, 11.

The Prize Medal, (value Six Dollars) has been determined to James Temple, *New-York.*

ARTICLE IX.

A short Disquisition, concerning the Definition, of the word Power, *in Arithmetic and Algebra.*

By G. Baron.

IN the following paper, I contend, that, in the writings of the modern mathematicians, the *powers of numbers*, are inaccurately defined; and that in consequence of this inaccuracy, certain conclusions, in the higher branches of the mathematics, evidently true in their own nature, appear as mysterious and incomprehensible paradoxes. The definition, to which I allude, has generally been expressed, in words to the following effect: *the powers of any number, are the successive products, arising from the continual multiplication of that number, into itself.* Now, if this definition, be considered as perfectly accurate, it must incontrovertibly follow, that, the first power of any number, is produced by multiplying that number, by itself;—that the second power, of a number, is produced, by multiplying the first power, of that number, by the number itself;—that the third power, of a number, is produced, by multiplying the second power, of that number, by the number itself; &c. &c.

Hence $5 \times 5 = 25$, the first power of 5,
$25 \times 5 = 125$, the second power of 5,
$125 \times 5 = 625$, the third power of 5, &c. &c.

The first, second, third, &c. powers of any number, might be conveniently expressed thus, 5^1, 5^2, 5^3, &c; where $5^1 = 5 \times 5 = 25$, the first power of 5; $5^2 = 25 \times 5 = 125$, the second power of 5; $5^3 = 125 \times 5 = 625$, the third power of 5; &c. &c.

In the same manner, the first, second, third, &c. powers of any number x, might be represented, by x^1, x^2, x^3, &c, in which $x^1 = x \times x$, $x^2 = x^1 \times x$, $x^3 = x^2 \times x$, &c. &c. All this is, evidently, in exact conformity, with the common definition, of powers. But it is well known that the word power is not, and cannot be, used in this sense; for, in arithmetic and algebra, every number, is necessarily considered, as equal to the first power, of itself: and hence, since it is impossible, according to the common definition, for any number to be equal to its own first power; it follows, that, this definition is erroneous. Further, it is also well known, that, the product resulting from any number, multiplied by itself, must be considered as the second power, of that number: but according to the common definition, it is only the first power; and hence this definition is again erroneous: in a similar manner, the fallacy of the same definition, might be shown when applied to any other power whatever. Since, therefore, it has been shewn, that, the common definition of powers is universally false, it becomes absolutely necessary, to expel it from the mathematical sciences, and to establish another that shall be truly correct. For the certainty and evidence of these sciences, depend essentially on the precision and accuracy, with which the various terms are defined: every mathematician knows, that, the limits, or bounds of a definition, form the true partition line between science and mystery; and that the moment we pass these limits, we must

inevitably, enter the incomprehensible and visionary regions.

From a careful survey, of the application of powers, in every branch of the mathematical sciences, it has long appeared to me, that the following definition, is universally correct. Viz. *The powers of any number, are the successive products, arising from unity, continually multiplied, by that number.* According to this definition, unity multiplied by any number, produces the first power of that number; —the product arising, from the first power of any number, multiplied by the number itself, is the second power of that number;—the product resulting, from the second power of any number, multiplied by the number itself, is the third power of that number; &c. &c.

Hence $\underline{1} \times 5 = 5$, the first power of 5,

$\underline{1 \times 5} \times 5 = 25$, the second power of 5,

$\underline{1 \times 5 \times 5} \times 5 = 125$, the third power of 5,

$\underline{1 \times 5 \times 5 \times 5} \times 5 = 625$, the fourth power of 5, &c. &c.

Let now 5^1 denote the first power of 5, or 1×5; 5^2, the second power of 5, or $1 \times 5 \times 5$; 5^3, the third power of 5, or $1 \times 5 \times 5 \times 5$; &c. &c. where the small figures 1. 2. 3. &c. are called the exponents of the first, second, third, &c. powers. In like manner, 7^1, 7^2, 7^3, &c. will represent the first, second, third, &c. powers of 7; and x^1, x^2, x^3, x^4, &c. the first, second, third, fourth, &c. powers of any number x. It may now be necessary to demonstrate the two following propositions.

PROP. I.

The first power of any number, is equal to that number itself.

For, let x represent any number, whatever; then from the definition, and the nature of numbers, $x^1 = 1 \times x = x$.

PROP. II.

The second power of any number, is equal to the product, arising from that number multiplied by itself.

For, by the definition, $x^2 = \overline{1 \times x} \times x = x^1 \times x$; and by Prop. I. $x^1 = x$; consequently $x^2 = x \times x$.

COROLLARY.

Hence $x^3 = \overline{1 \times x} \times x \times x = x^2 \times x = x \times x \times x$, $x^4 = x^3 \times x = x \times x \times x \times x$, &c. &c. And hence any product, arising from the continual multiplication of a number, into itself, is a power of that number; and the number of factors used, in such continual multiplication, is equal to the exponent of the power, so produced.

Thus far, every conclusion, drawn from the propounded definition, agrees, exactly, with the common usage of every mathematical author of the present age. Let us, therefore, next inquire, whether the same definition, will not lead us to a clear and intelligible solution, of the mysterious paradoxes, resulting from the common definition, when applied, to what is denominated, the *nothingth* powers of numbers. The word nothing, in a mathematical sense, signifies the privation of some property, which, *necessarily* exists, in the constitution of any thing, considered as the subject of number, or of magnitude. This may be clearly illustrated from the nature of solidity: for, length, breadth, and thickness, necessarily exist together, in the constitution of a solid; and if thickness be abstracted, length and breadth will remain; but

the magnitude of this solid, when thus deprived of thickness, will be *nothing*. The word *nothing*, here applied to solidity, evidently signifies, the privation of a property which must necessarily exist, in the constitution of a solid: and hence this *nothing* does not affirm absolute nonentity; for the remaining length and breadth, constitute a superficies; a thing perfect in its own nature, though destitute of solidity.

The learned Mr. Emerson, in the scholium to the 73d* problem of his Algebra, very justly observes, "that, nothing in a mathematical sense, "never signifies absolute nothing; but always "nothing in relation to the object under consider-"ation." If now, this mathematical idea of *nothing*, be applied to our definition, it will be evident that unity, is the nothingth power of any number whatever. For let x denote any number; then by the definition, $1 \times x$, is the first power of x. Hence then, a unit multiplied by x, must, necessarily constitute the first power of any number x; and therefore if the multiplication by x, be abstracted from this first power of x, by means of division; the power will become nothing but the unit will remain: for $\frac{x^1}{x} = \frac{1 \times x}{x} = 1$, and hence it is plain that $x^0 = 1$, when x represents any number whatever. But since the *number* x, is here unlimited with regard to greatness, it follows, that, the nothingth power of an infinite number is equal to a unit.

What, has hitherto been advanced, relates to pure numbers; and it now remains, to apply our

* Mr. Emerson wrote this problem in the *English language*, but it has lately been translated into *nonsense*, by Jared Mansfield of Connecticut, and published in a wonderful work, which this translator calls Mathematical Essays. The indefinitely small *nothings* of Connecticut, are infinitely great absurdites, in the regions of science and common sense.

definition of powers, to abstract quantities. By a pure number, I mean, a definite plurality, or a determinate multitude, of abstract and indivisible units; and by abstract quantity, I understand, magnitude expresssed in numbers, by means of some known measure, considered as a unit, indefinitely divisible. From a certain corresponding analogy, between the quantities of geometrical rectangles, and the products of abstract numbers, mathematicians have linked the former with the latter, subjected quantity, to all the operations of numbers, and proved, that, every proposition, that is true of abstract numbers, is also true of abstract quantities. Hence, then, since, our definition, was universally accurate, in its application to abstract numbers; it will likewise be universally correct, when applied, to abstract quantities. But, in order, to pursue the application of our definition, to quantity in the ultimate extremity of smallness, let us suppose x to represent any fractional quantity; or in other words, let x denote any magnitude, expressed in numbers, by means of some part of its measuring unit: then by the definition $x^1 = 1 \times x$. Let now this multiplication by x, be abstracted; and for the reasons heretofore advanced, we have $x^\circ = 1$. Now since x here represents a fractional quantity, independent of any limitation, in respect to smallness; we may therefore suppose x, by means of continual diminution, or decrease, to pass from its present value, through every degree of smallness, until it become *nothing*; then it will be evident, that, during this diminution or decrease of x, x° will continue equal to an invariable unit; and that *precisely at the instant*, when x becomes *nothing*, x°, or $0^\circ = 1$. We have therefore proved, immediately from our definition, that, the nothingth power of every number and quantity, is equal to a unit; and, that,

$0^0 = 1$: the truth of which propositions is well known; and hence the definition itself, is universally correct. Indeed, when powers are thus defined, these propositions are self-evidently true, independent of the demonstrations here given: but on the contrary, when powers are considered, as generated according to the common definition; the same propositions, (notwithstanding the demonstrations of modern analysts,) must inevitably appear, as incomprehensible paradoxes. For, if powers be the successive products, resulting from the continual multiplication of any number into itself; such multiplication, must be essential in the production of a power; and where there has been no such multiplication, there can be no power. So long, therefore, as the common definition is retained, it will be absurd to consider a number as the first power of itself. For by that definition any number x multiplied by itself, gives the first power of $x = x^1 = x \times x$; and when this multiplication by x, is abstracted we have $x^0 = x$, which is a manifest absurdity, and utterly repugnant to the well known nature of logarithms. Further, since in adhereing to the same definition it has been shewn, that, $x^0 = x$; it would therefore be impossible for x^0, to be universally equal to a unit. But we know, from the nature of logarithms, and also, from other principles, that x^0, is actually and universally equal to a unit; and hence we infer, that, the inexplicable mysteries, which have long been attributed, to the nothingth powers of numbers and quantities; originated, in a wrong definition of the word *power*.

From what has here been said on the subject, it is plain, that, the powers of any number, form a geometrical series from unity, whose ratio is the number itself; and that the exponents of such a

series of powers, form an aritmetical series, from 0, whose common difference is a unit: or in other words, that, the exponents of the powers of any number, are a certain system of the logarithms of their corresponding powers. And hence it follows, that, all the operations of numbers, performed by logarithms, may in a like manner, be done of powers, by means of their exponents. Also, since $x^{\circ} = 1$, whatever be the value of x; of consequence, in every system of logarithms, the logarithm of $1 = 0$.

ARTICLE X.

NEW QUESTIONS to be answered in the next Number.

I. QUEST. 19. *by Niel Gray, New-York.*

Out of an annuity of 1000 dollars per annum, for 10 years, the first payment being due one year hence, the owner desires to know how much he may spend a year; so that his annual savings, with the simple interest arising therefrom, at 7 per cent. per ann. may, at the expiration of this annuity, amount to a sum whose interest at 7 per cent. per ann. shall be equal to the yearly expenditure required.

II. QUEST. 20. *by Alexander Walsh, New-York.*

It is required to determine whether 30 horses can be put into 7 stalls; so that in every stall there may be, either a single horse, or an odd number of horses.

III. QUEST. 21. *by Ebenezer R. White, Danbury Connecticut.*

Given $x = \sqrt[r]{a^m + b^n}$, and $y = \sqrt[r]{a^m - b^n}$, supposing $r = 5$, $m = 9$, $n = 7$, $a = 684{\cdot}588$ and $b = 24{\cdot}5632$; required a general rule, for finding the values of x and y, by a table of common logarithms.

IV. QUEST. *by the Rev. T. P. Irving, Newbern, North Carolina.*

In inches, the head of a Pamtico fish,
Which makes, on the board, a delectable dish,
The thirty-five thousand two hundredth, will be,
Of years circumvolving, (the learn'd all agree,)
'Twixt the century current, and when in amaze,
The world on a spring sempiternal, shall gaze.
It's body bisected, that's half cut asunder,
You see I'm Hibernian, but smart's, I'll not blunder;
I mean, that, one half of its body's whole length,
Conjoin'd to those inches, by *plus's* whole strength,
Its 'ail will express; while its body if sought,
Is more the head, more the tail, eke more a naught.
Now tell me ye skill'd in symbolical art,
The length of this fish, and each specified part

V. QUEST. 23, *by Diarius Yankee, Bunker's Hill.*

A gentleman a garden has,
In area, just an acre;
In form a circle, on whose verge,
Was sat *John Bull*,—a quaker:
Whose head oft-times, it must be owned,
Was teeming with strange *fangles*
And horticulture studied less,
Than circles, lines, and angles.—
Here, master, look thee, *John* exclaim'd,
Three trees thou seest—I blunder
Two trees I mean—I would have three;
Pray what excites thy wonder?
The third I'd plant outside the fence,
For this thou seest grows *in* it;
Just 90 feet from that whose boughs
So much frequents the linnet;
But so that whence soe'er thou view'st
Them from the garden's ring,
They'll equi-distant seem—if not—
May I ne'er be a king!
John's master knowing what he meant
Consent at once was granted,
Pray shew us where—I know you can
The tree by John was planted.

VI. QUEST. 24. *by the Rev. J. Blackburn, Cambridge, England.*

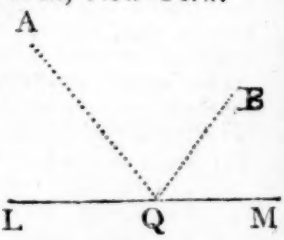
If a circle ABDC, circumscribe an equilateral triangle ABC; the straight line AD is equal to the sum of the two straight lines BD and DC: required a demonstration.

VII. QUEST. 25. *by Robert Adrain, York-Town, Pennsylvania.*

If any quadrilateral ABCD, be inscribed in a circle, and the diagonals AC, BD be drawn; it will be, as the sum of the rectangles BA . AD, and BC . CD, is to the sum of the rectangles AB . BC, and AD . DC, so is the diagonal AC, to the diagonal BD. Required a demonstration.

VIII. QUEST. 26. *by G. Baron, New-York.*

A ray of light issues from a given point A, and is constantly reflected to another given point B, by a plane speculum LQM, which moves parallel to itself; required the locus of all the points Q.



IX. QUEST. 27, *by Diarius Yankee, Bunker's Hill.*

The cavity of our chimney, is an upright parallelopipedon, the diagonal of whose base is 60 inches; and the height of the lower side of the lintel, above the plane of the floor is 40 inches. What is the length of the longest inflexible stick that can be put up this chimney?

X. PRIZE QUEST. 28. *by Wm. Green, New-York.*

(The author of the best solution, to this Question, shall receive a handsome silver Medal, value six Dollars.)

St. John's, in Newfoundland is in lat. 47° 32′ N. long. 52° 26′ W. Cape Finisterre, in Spain, is in lat. 42° 51′ 52″ N. long. 9° 17′ 10″ W. and Cape Barbas, in Africa, is in lat. 22° 15′ 30″ N. long. 16° 40′ W. Now there is a certain point, (on the same hemisphere of the earth,) which, on the arcs of great circles, is equally distant, from each of these three places; and it is required to determine this distance, the bearings of the three places from the point, the latitude and longitude of the point, and the courses and distances from the same point to each of the same three places.

ARTICLE XI.

Answers to the Questions proposed in Article X.

I. Quest. 19. *Answered by Thomas Whittaker, Harrisburg, Pennsylvania.*

PUT a=the annuity=1000 dollars, t=the time of continuance=10 years, r=the interest of 1 dollar for a year=·07, and $a-x$=the expenditure required. Then, by the common formula, and the nature of the question; $\frac{\overline{t-1} \times r+2}{2} \times trx = a-x$. Whence $x = \frac{2a}{\overline{t-1} \times tr^2+2tr+2} = 520{\cdot}6977$ dollars, and $a-x=$ 479·3023 dollars, the annual expenditure sought.

II. Quest. 20. *Answered by James M'Ginness, Middletown, Pennsylvania.*

The question is, "Whether 30 horses can be put into 7 stalls, so that in every stall there may be either a single horse or an odd number of horses." Let $a+b+c+d+e+f+g=30$; then will $\overline{a+1}+\overline{b+1}+\overline{c+1}+\overline{d+1}+\overline{e+1}+\overline{f+1}+\overline{g+1}=37$, an odd number. Now, by the question, each of the seven letters a, b, c, d, e, f and g, is either a unit or an odd number: consequently $a+1$, $b+1$, $c+1$, $d+1$, $e+1$, $f+1$, and $g+1$, are even numbers; and, by Prop. 21, book 9, Euclid, their sum (37) is also an even number. But this last conclusion is evidently absurd; and therefore the horses cannot be put into the stalls so as answer the conditions of the question.

The same answered by the Rev. Thomas P. Irving, Newbern, North-Carolina.

Sirs, Mr. Walsh, I think, must own,
One horse is useless lumber;
Unless he'll make three *odds* combin'd
Produce an even number.

OTHERWISE.

I think Mr. Walsh *Alexander*, must own
That *Bucephalus* sleeps out o' doors;
Unless he has skill to rectangle three threes,
That shall do what is done by four fours.

III. QUEST. 21. *Answered by Ebenezer R. White, Danbury, Connecticut.*

After correcting a typographical error, which, in this question, made $n=7$, instead of $n=17$; let the first of the given equations, $x=\sqrt[r]{a^m+b^n}$, be transformed to $\sqrt[r]{\left(\frac{a^m}{b^n}+1\right)\times b^n}$: from which, the first of the general rules required is self-evident. Hence the value of x will be found as follows:

$$\begin{aligned}
\log.\ a=\log\ 684{\cdot}588 &= 2{\cdot}8354293\\
m &= 9\\
\log.\ a^m &= 25{\cdot}5188637
\end{aligned}$$

$$\begin{aligned}
\log.\ b=\log.\ 24{\cdot}5632 &= 1{\cdot}3902849\\
n &= 17\\
\log.\ b^n &= 23{\cdot}6348433\\
&\quad 25{\cdot}5188637\\
&\quad 1{\cdot}8840204
\end{aligned}$$

$$\log.\frac{a^m}{b^n} = \log. 76{\cdot}5633 = 1{\cdot}8840204$$

$$\log. 77{\cdot}5633 = 1{\cdot}8896563$$

$$\log. b^n = \ldots\ldots\ldots 23{\cdot}6348433$$

$$r=5)\overline{25{\cdot}5244996}$$

$$\log. x = \log. 127321 = 5{\cdot}1048999$$

Hence $x=127321$ nearly; and by a similar process we obtain the second general rule; from which is found, $y=126657$ nearly. In the same manner, the value of a binomial surd may often be easily found, when the common methods of calculation would render the operation exceedingly troublesome.

IV. QUEST. 22. *Answered by Diarius Yankee, Bunker's Hill.*

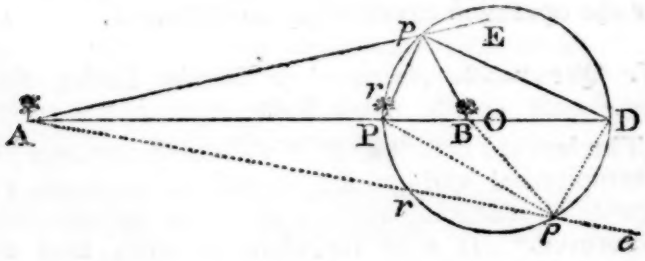
The learned do not expect the commencement of a sempiternal spring: for, in nature, no cause, cacapable of producing such an effect, has hitherto been discovered.* It may therefore be said, that the learned all agree, that, should an eternal spring commence, at some future period, the distance of that period from the present century is now unknown. Hence the first six lines of this question may be considered as an ingenious enigma, signifying that the length of the fish's head is not given. Let, now, $2y$=length of the body, and x=the length of the head; then by the question $y+x$=the length of the tail, and $3y+2x$=the length of the fish. Also by the question $2y=y+2x+0$; whence $y=2x$; and consequently if $x=1$, $y=2$, $y+x=3$, $2y=4$, and $3y+2x=8$; or the length of the head, tail, body and fish are as 1, 3, 4 and 8. Therefore the question admits of an indefinite number of answers.

* See Prop. 34, Emerson's Centripetal Forces.

V. QUEST. 23. *Answered by G. Baron, New-York.*

LEMMA.

If in a straight line A B, drawn on the plane E A D, a point P be taken between A and B, and nearer to B than to A; the locus of all the points P, p, on the same plane, is a circle, when the straight line p P always bisects the rectilineal angle A p B: and the centre of this circle is in the straight line A B produced towards D.



DEMONSTRATION.

Let the point P, in moving from P to p, describe the line P r p. Then, since by hypothesis the straight line p P bisects the rectilineal angle A p B; A P : P B : : A p : p B, by prop. 25, book 2, Emerson's Geom. 2d edit. Again, by hypothesis, A P is greater than P B; and therefore the increment of A p is always greater than the cotemporary increment of p B: of consequence A p—p B continually increases. But when A p—p B becomes equal to A B, or when A p =A B+p B; the triangle A p B vanishes, the point p coincides with D, and the straight lines A p and p B respectively coincide with, and become equal to A D and B D. Hence A P : P B : : A p : p B : : A D : D B;

and therefore, by the prop. above quoted, the straight line p D bisects E p B, the exterior angle of the triangle A p B. Lastly, since p P bisects the angle A p B, E p D+A p P=P p D=a right angle; and consequently the curve P r p D is a semicircle, whose centre O is in A B, produced towards D. *Q. E. D.*

Corollary.

When A P=P B, the triangle A p B will be isoceles; and the locus of the point P will be a straight line. But if A P be less than P B, the locus of P is a circle, whose centre is in B A, produced beyond A.

N. B. The demonstration of this lemma might have been inferred from prob. 13, const. geom. probs. Simpson's Geom. 4th edit. and the reverse might have been drawn from prop. 23, book 4, Emerson's Geom. 2d edit.

Solution.

By the data of the question, and the principles of perspective, a straight line, drawn from any point in the circular fence to the middlemost tree, must bisect the angle formed by straight lines drawn from the same point to each of the other two trees. But, since one of the trees grows in the fence, the third tree must be planted in a straight line with the first and second; otherwise, a straight line, drawn from some point in the fence, through the middlemost tree, might also pass through one of the other trees, without passing through the third; which is contrary to the conditions of the question. Now, by the above lemma, if B, P, and A, represent the places of the three trees, in a straight line A B, P r p D p r will represent the circular fence, whose centre O is in

A B, produced beyond B: for, by the question, the third tree A was to be planted outside the fence. Also by the lemma D B : A D : : P B : P A; and by a theorem in proportion D B—P B=2 B O : A D—P A=2 O P : : P B : P A, or O B : O P : : P B : P A. Again, by the question and a well known theorem, the radius O P=$\sqrt{\frac{1\times660\times66}{3\cdot14159, \&c.}}$=117·75219 feet: and since P B is given=90 feet, O B=O P—P B=27·75219 feet. Hence, in the four proportionals, the three first terms O B, O P, and P B, are given to find the fourth P A=381·8692 feet: and consequently the place of the third tree, A, is known.

COROLLARY.

O A is a third proportional to O B and O P: for O B : O P : : P B : P A : : O B+P B=O P : O P+P A=O A.

VI. QUEST. 24. *Answered by the Proposer, the Rev. J. Blackburn, Cambridge, England.*

Because equilateral triangles are equiangular, and the angles, at the circumference of a circle, standing on the same arc, are equal one to another, the angles B D A=B C A=A B C=A D C: and hence A D bisects the angle B D C. In C D produced, take D F=D B, and join B F. Then the angle D B F =D F B by 5, 1, Euc.=A D C by 32, 1, Euc. for it has been shown A D, that bisects the angle B D C. But the angle B D F=B A C by 22, 3, Euc.=A B C =A D C=D B F=D F B, by what has already been

proved; and therefore the triangle D B F is equiangular, and consequently equilateral. Further, since the angle D B F=A B C, the angle C B F=A B D. Lastly, in the triangles A B D and C B F, the sides A B and B D are respectively equal to the sides C B and B F; and the angles contained by these sides are also equal; therefore, by 4, 1, Euc. A D=C F, by construction=C D+D B. *Q. E. D.*

OTHERWISE.

Since the angles at the circumference of a circle, standing on the same arc, are equal; and since equilateral triangles are equiangular; the angle A B C= A D C=B C A=A D B; of consequence, A D bisects the angle B D C. Hence, by 3, 6, Euc. B E : E C : : B D : D C; and by composition, B C : E C : : B D+ D C : D C. But by similar triangles A C=B C : E C : : A D : D C; therefore, by 11, 5, Euc. B D+D C : D C : : A D : D C. Hence by 9, 5, Euc. B D+D C =A D. *Q. E. D.*

OTHERWISE.

Because A B D C is a quadrilateral, inscribed in a circle, B D×A C+D C×A B=A D×B C, by prop. D, book 6, Simson's Euclid. But A C=A B=B C, by hypothesis; and consequently BD+DC=A D. *Q. E. D.*

This property of an equilateral triangle, inscribed in a circle, has been long known, and was re-proposed in this work, with a view of inducing the young American geometrician to exercise himself frequently in the discovery of new demonstrations to propositions already established. The first and second of the above methods of demonstration are presumed to be new; but the third may be found in various mathematical treatises.

A B, produced beyond B: for, by the question, the third tree A was to be planted outside the fence. Also by the lemma D B : A D : : P B : P A; and by a theorem in proportion D B—P B=2 B O : A D—P A=2 O P : : P B : P A, or O B : O P : : P B : P A. Again, by the question and a well known theorem, the radius $O P=\sqrt{\frac{1\times660\times66}{3\cdot14159, \&c.}}=117\cdot75219$ feet: and since P B is given=90 feet, O B=O P—P B=27·75219 feet. Hence, in the four proportionals, the three first terms O B, O P, and P B, are given to find the fourth P A=381·8692 feet: and consequently the place of the third tree, A, is known.

COROLLARY.

O A is a third proportional to O B and O P: for O B : O P : : P B : P A : : O B+P B=O P : O P+P A=O A.

VI. QUEST. 24. *Answered by the Proposer, the Rev. J. Blackburn, Cambridge, England.*

Because equilateral triangles are equiangular, and the angles, at the circumference of a circle, standing on the same arc, are equal one to another, the angles B D A=B C A=A B C=A D C: and hence A D bisects the angle B D C. In C D produced, take D F=D B, and join B F. Then the angle D B F =D F B by 5, 1, Euc.=A D C by 32, 1, Euc. for it has been shown A D, that bisects the angle B D C. But the angle B D F=B A C by 22, 3, Euc.=A B C =A D C=D B F=D F B, by what has already been

proved; and therefore the triangle D B F is equiangular, and consequently equilateral. Further, since the angle D B F=A B C, the angle C B F=A B D. Lastly, in the triangles A B D and C B F, the sides A B and B D are respectively equal to the sides C B and B F; and the angles contained by these sides are also equal; therefore, by 4, 1, Euc. A D=C F, by construction=C D+D B. Q. *E. D.*

OTHERWISE.

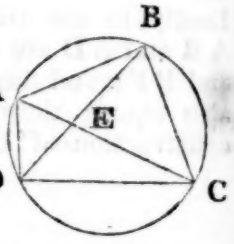
Since the angles at the circumference of a circle, standing on the same arc, are equal; and since equilateral triangles are equiangular; the angle A B C= A D C=B C A=A D B; of consequence, A D bisects the angle B D C. Hence, by 3, 6, Euc. B E : E C : : B D : D C; and by composition, B C : E C : : B D+ D C : D C. But by similar triangles A C=B C : E C : : A D : D C; therefore, by 11, 5, Euc. B D+D C : D C : : A D : D C. Hence by 9, 5, Euc. B D+D C =A D. Q. *E. D.*

OTHERWISE.

Because A B D C is a quadrilateral, inscribed in a circle, B D×A C+D C×A B=A D×B C, by prop. D, book 6, Simson's Euclid. But A C=A B=B C, by hypothesis; and consequently BD+DC=A D. Q. *E. D.*

This property of an equilateral triangle, inscribed in a circle, has been long known, and was re-proposed in this work, with a view of inducing the young American geometrician to exercise himself frequently in the discovery of new demonstrations to propositions already established. The first and second of the above methods of demonstration are presumed to be new; but the third may be found in various mathematical treatises.

VII. QUEST. 25. *Answered by William Lenhart, York-Town, Pennsylvania.*



Because the triangles ABE and DEC, as also AED and BEC, are similar; AB : DC : : BE : CE, and BC : AD : : CE : DE: consequently AB×BC : DC×AD : : BE : DE; and by composition, AB×BC+DC ×AD : DC×AD : : BE+DE=BD : DE : : DC×BD : DC×DE. In the same manner it may be proved that AB×AD+BC×DC : BC×DC : : AC : CE : : DC×AC : DC×CE. Hence by alternation, &c.

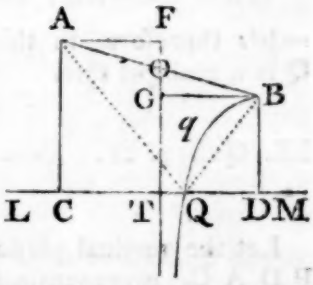
AB×BC+DC×AD : DC×BD : : AD : DE, and
AB×AD+DC×BC : DC×AC : : BC : CE.

But by similar triangles AD : DE : : BC : CE; therefore, by equality, AB×BC+DC×AD : DC ×BD : : AB×AD+DC×BC : DC×AC: and by alternation, &c. AB×BC+DC×AD : AB×AD+ DC×BC : : BD : AC. *Q. E. D.*

COROLLARY.

Hence the diagonals of any quadrilateral, inscribed in a circle, may be easily found when the four sides are given: for, by the above theorem, the ratio of the diagonals is given; and their rectangle is also given by prop. D, book 6, Simson's Euclid.

VIII. QUEST. 26. *Answered by John D. Craig, Philadelphia.*



Bisect the straight line AB in O; on LM let fall the perpendiculars AC, OT and BD; and draw BG and AF, at right angles, to OT and TO produced. Then, by the question, and a well known optical principle, the angle AQL=BQM; and, of consequence, the right angled triangles ACQ and BQD are similar: hence AC : BD : : CQ : QD. But since the right angled triangles AFO and BGO are evidently similar and equal in all respects, OF=OG, and AF=BG=CT=TD; also AC=TO+OG, BD=TO—OG, CQ=BG+TQ, and QD=BG—TQ: therefore TO+OG : TO—OG : : BG+TQ : BG—TQ; from which TO : OG : : BG : TQ. Hence the curve B*q*Q, the required locus, is an equilateral hyperbola; to which OT is an asymptote.

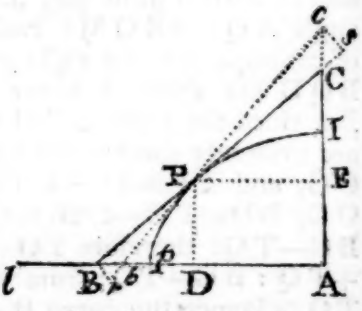
The same answered by John Craggs, near Richmond, Virginia.

From the given points A and B draw AC and BD, at right angles to LM. Put AC—BD=*a*, DC=*d*, BD=*x*, and DQ=*y*; then AC=*a*+*x*, and CQ=*d*—*y*. Also, from the conditions of the question, and a principle in optics, the angle AQL=BQM; and consequently the right angled triangles ACQ and BDQ are similar. Hence AC : BD : : CQ : QD, or *a*+*x* : *x* : : *d*—*y* : *y*; and therefore, by composition, $a+2x : x : : d : y=\frac{dx}{a+2x}$, the equation of the curve B*q*Q, the locus required.

COROLLARY.

When AC=BD; $a=o$; and $y=\frac{dx}{a+2x}$ becomes $y=\frac{1}{2}d$: therefore, in this case, the locus of the point Q is a straight line.

IX. QUEST. 27. *Answered by James Temple, New-York.*



Let the vertical plane P D A C, representing the diagonal section of the chimney, be perpendicular to A B, the plane of the horizontal floor. Then, by the question, A D=60 inches, is the diagonal of the chimney's base; D P=40 inches, is the height of the lintel above A B, the plane of the floor; and A C is perpendicular to A B and parallel to D P. Through the point P draw P E parallel to A B; and let the straight line C P B represent the required length of the stick. Again, through the point P draw the straight line bc indefinitely near a coincidence with B C; and on the centre P, with the radii P c and P B, describe the small arcs cs and B r. Now, by similar triangles, P D : B D : : B r : $br=\frac{B\,r\times B\,D}{P\,D}$; and B D : P D : : P E : C E : : cs : $Cs=\frac{cs\times P\,D}{B\,D}$. But, by the question, C B is a maximum; and therefore it is well known that br, the indefinitely small decrement of P B, is equal to C s, the indefinitely small cotemporary increment of P C: con-

sequently $\frac{Br \times BD}{PD} = \frac{cs \times PD}{BD}$, from which $\overline{PD}^2 = \overline{BD}^2 \times \frac{Br}{cs}$. Further, by similar sectors, $cs : Br ::$ $Pc : PB ::$ (by similar triangles) $PE : BD$; whence $\frac{Br}{cs} = \frac{BD}{PE}$. Therefore, $\overline{PD}^2 = \overline{BD}^2 \times \frac{BD}{PE} = \frac{\overline{BD}^3}{PE}$, from which $BD = \sqrt[3]{\overline{PD}^2 \times PE} = 45{\cdot}7886$, and $BA = BD + DA = 105{\cdot}7886$. Lastly, in the right angled triangle PDB, $PB = \sqrt{\overline{BD}^2 + \overline{PD}^2} = 60{\cdot}8$; and hence, by similar triangles, $BD : BA ::$ $BP : BC = 140{\cdot}47$ inches, the required length of the stick.

The same answered by R. Tagart, New-York.

The figure representing as in the above solution; let lA be the stick whose length is required. Imagine the end A to be drawn up the vertical straight line AC; then a point p, which must necessarily touch the lintel at P, will describe the curve pPI. But by p. 49 and 50, Math. Correspondent, this curve is an ellipse, to which the stick, in the position BC, is evidently a tangent at P. Let now $PD = AE = a = 40$ inches, $DA = PE = b = 60$ inches, and $BD = x$; then from the nature of the ellipse $AD : Ap = CP ::$ $CP : AB$: whence $\overline{CP}^2 = AD \times AB = b^2 + bx$. Also, by similar triangles, $BD : PD ::$ $PE : CE = \frac{ab}{x}$; and therefore $\overline{CP}^2 = \overline{PE}^2 + \overline{CE}^2 = b^2 + \frac{a^2 b^2}{x^2} =$ $b^2 + bx$. Hence $x = BD = \sqrt[3]{a^2 b} = 45{\cdot}7886$ inches; and consequently $BC = l$A, the required length of the stick, is easily found $= 140{\cdot}47$ inches.

X. Prize Quest. 28. *Answered by Thomas Maughan, Quebec.*

Let I represent St. John's, F Cape Finisterre, B Cape Barbas, and I P, F P and B P, their meridians, meeting in P, the north pole of the earth. With the arcs of great circles, I F, F B and B I, join the points I, F and B; and bisect I F and F B in the points A and C. At right angles to I F and F B draw the great circular arcs A p and C p; which will intersect each other in p, the point required in the question. For from p to I, F, and B, draw the arcs of great circles p I, p F, and p B; then in the triangles I A p and p A F, right angled at A, I A = A F, and A p is common to each triangle; of consequence, the third sides p I and p F are equal. In the same manner it may be proved, from the right angled triangles F C p and p C B, that p B = p F; whence p I = p F = p B, and therefore p is the point whose latitude and longitude are sought. Now, from the question, the complements of the three given latitudes are I P = 42° 28′, F P = 47° 08′ 08″, and B P = 67° 44′ 30″; and the differences of the three given longitudes are the angles I P F = 43° 08′ 50″, I P B = 35° 46′, and B P F = 7° 22′ 50″. Hence, then, in each of the three triangles, I P F, I P B and B P F, two sides and the included angle are given; from which, by means of versed sines, &c. the third sides are easily found: viz. I F = 30° 21′ 35·3″, I B = 38° 08′ 50·6″, and B F = 21° 30′ 08·9″. Also, in each of the same three triangles, the three sides and one angle are now known; from which may be found the angles P I F = 82° 38′, P F I = 65° 59′ 54″, B I P = 118° 52′ 02″, and P F B = 161°

04′ 26″: consequently the angle F I B=P I B—P I F =36° 14′ 02″, and the angle I F B=P F B—P F I= 95° 04′ 32″. Again, in the triangle I F B the three sides and two angles are known; and the third angle F B I is found=54° 35′ 33″. Further, from what has been done, it appears that the angle I F B (=95° 04′ 32″) is greater than F B I+F I B (=90° 49′ 35″); and since p I=p F=p B, the angle p F I=p I F; and the angle p F B=p B F: consequently p F I+p F B=I F B =p I F+p B F, is known to be greater than B I F+ F B I; and therefore the point p is without the triangle I F B. But since p I=p B, the angle p I B= p B I; and hence the angle p I B=$\frac{\text{I F B—B I F—I B F}}{2}$ =2° 07′ 29″. Now, in the right angled triangle I A p are given the angle A I p=B I F+p I B=38° 21′ 31″, and the side A I=$\frac{1}{2}$ I F=15° 10′ 47·6″; from which is found the hypotenuse p I=p F=p B=19° 05′ 09·2″, the required distance on a great circle=1145·15 geographical minutes. Also in the triangle P I p are given the sides P I and I p, and the included angle P I p=P I F+A I p=120° 59′ 31″; to find the third side P p=54° 18′ 25″=the complement of the latitude of p, the angle I P p=20° 11′ 25″=difference of longitude of I and p, and the angle P p I=the bearing of I from p, N. 45° 27′ 01″ W. hence the latitude of p is 35° 41′ 35″ N. and its longitude is 32° 14′ 35″ W. Again, in the triangle B p P, the sides P B and p B, and the angle p P B=B P I—p P I are given; to find the angle P p B=the bearing of B from p, N. 130° 32′ 02″ E. or S. 49° 27′ 58″ E. In the same manner, from the triangle P p F, the bearing of F from p is found to be N. 60° 57′ 49″ E. Lastly, from the given latitudes and longitudes of p, I, F, and B, by the principles of Mercator's sailing, are found the courses from p to I, F, and B,=N. 51° 45′ 43″ W. N. 67° 59′ 34″ E. and S. 45° 18′ 00″ E. and the respective dis-

tances measured on these courses, 1147·81, 1148·27, and 1145·99 geographical minutes: the distance from *p* to I, to F, or to B, measured on a great circle, was found=1145·15 minutes. This question required the courses and distances from *p* to I, F, and C; but these found above are the least courses with their concomitant distances. It is, however, well known to mathematicians, that an infinite number of courses and distances may be found, which will carry a vessel from any one given place to another not in the same latitude; and that the greatest of such courses is indefinitely near 90°, having its corresponding distance indefinitely great. This will necessarily be thought a *strange doctrine* by the deluded disciples of *Bowditch* and *Blunt*, who are taught to believe that "plain sailing considers the earth as an extended plane, on which the meridians are parallel one to another"!! But to such people I have only to recommend the candid perusal of a small pamphlet, written by G. Baron, entitled, "Exhibition of the genuine Principles of common Navigation, with a complete Refutation of the false and spurious Principles, ignorantly imposed on the Public in the New American Practical Navigator."

ACKNOWLEDGEMENTS, &c.

An accurate list of the names and places of residence of the ingenious gentlemen who have given true solutions to questions proposed in No. III. The figures annexed to the names refer to the questions answered by each, as numbered in the work.

R. Adrain, York-Town, Penn.	1, 2, 3, 4, 5, 6, 7, 8, 9, 10.
G. Baron, New-York,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10.
Rev. J. Blackburn, Cambridge, Eng.	6.
John Capp, Harrisburg, Penn.	1, 2, 6.
John Craggs, near Richmond, Virg.	1, 2, 3, 4, 6, 7, 8, 9, 10.

John D. Craig, Philadelphia,	1, 2, 4, 5, 6, 7, 8, 9, 10.
Niel Gray, New-York,	1, 2, 3, 4, 5, 6.
William Green, New-York,	10.
Rev. T. P. Irving, Newbern, N. C.	1, 3, 4, 6, 7, 8.
Wm. Lenhart, York-Town, Penn.	1, 2, 5, 6, 7, 9, 10.
James M'Cormick, Carlisle, Penn.	2, 6.
J. M'Ginness, Middletown, Penn.	2, 6.
Thomas Maughan, Quebec,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10.
Ignace Romes, Middletown, Penn.	6, 7, 10.
John Smithis, Philadelphia,	1, 2, 5, 4, 6, 7, 8, 9, 10.
R. Tagart, New-York,	1, 2, 9.
James Temple, New-York,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10.
Wm. Thompson, Charleston, S.C.	1, 2, 3, 4, 6, 7, 8, 9, 10.
Alexander Walsh, New-York,	2.
Eben. R. White, Danbury, Con.	3,
T. Whittaker, Harrisburg, Penn.	1, 2, 4, 6, 7.
Diarius Yankee, Bunker's Hill,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10.
Titus Youle, Portland, Maine,	1, 2.
N. Young, Lexington, Kentucky,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10.

The Prize-Medal (value six dollars) has been awarded to *Thomas Maughan*, Quebec, who is requested to authorize some person to receive the same from Mr. *George Baron*, Mathematician, New-York.

☞ The Editors tender their grateful acknowledgements to the Scientific Society of New-York for presenting the prize-medal to Mr. Temple, in a public and honourable manner—to Dr. Buxton, for the excellent oration which he then delivered on the subject—and to the President and officers of the same Society, for their very handsome and obliging conduct on that occasion.

ARTICLE XII.

An Inquiry respecting the true Definition of Proportional Numbers. By G. BARON.

Proportion is the marrow of the mathematics, and the very soul of geometry and geometrical reasoning.—*Emerson.*

The modern mathematicians have written the highest encomiums on the doctrine of Proportion; but in what page of their works can we find the ge-

nuine nature of four proportional numbers clearly developed? Proportion is one of the most essential articles in the science of multitude: its application is infinitely more extensive and useful than all the other parts of arithmetic: and hence a competent knowledge of this subject is, to young arithmeticians, an indispensable requisite. But it is well known, that every branch of the mathematical sciences must be deduced from definitions which are precisely and universally correct; and therefore the present inquiry will appear to be expedient and necessary.

To begin this inquiry, we ask whether all arithmeticians agree that, in any four proportional numbers, either the first is equal to the second, and the third equal to the fourth; or the first is a multiple of the second, and the third a like multiple of the fourth; or the first is a part of the second, and the third a like part of the fourth; or the first is a multiple of a part of the second, and the third a like multiple of a like part of the fourth?* The answer must be, that this well known truth is incontrovertible, and universally admitted. Hence, then, it appears that, in numbers, there are four different species of proportionals; and that we are already furnished with true definitions of these species, enumerated as follows.

Definition of the first species.

Four numbers are proportional, when the first is equal to the second, and the third is equal to the fourth.

Definition of the second species.

Four numbers are proportional, when the first is a multiple of the second, and the third is a like multiple of the fourth.

* Here the reader is requested to peruse page 1, 2, and 3, Mathematical Correspondent.

Definition of the third species.

Four numbers are proportional, when the first is a part of the second, and the third is a like part of the fourth.

Definition of the fourth species.

Four numbers are proportional, when the first is a multiple of a part of the second, and the third is a like multiple of a like part of the fourth.

N. B. The above definitions are confined to *numbers** merely for the sake of brevity; but it is readily perceived, that all the species, except the fourth, will admit the *unit*.

Now, since the aggregate of the definitions of all the species will never define the genus, we are still unacquainted with the generical nature of four proportional numbers. We know, however, that proportionality must here consist of something common to, and necessarily existing in, each of the four species; but whatever this may be, it cannot depend on the divisibility of a unit: for this inquiry relates solely to abstract numbers, whose units are void of extension, and, consequently, indivisible. Again, since proportionality here consists of something common to, and necessarily existing in, each of its species, it must, therefore, be something common to, and necessarily existing in, each of the four following

Examples of proportional numbers.

1st species,	7	:	7	: :	13	: 13.
2d species,	12	:	4	: :	15	: 5.
3d species,	3	:	18	: :	2	: 12.
4th species,	16	:	12	: :	8	: 6.

* Numbers are composed of units, but a unit is not a number: if a book be said to consist of leaves, it is plain that a leaf is not a book.

Further, since each of these species is defined by means of a particular *relation* subsisting in common between the first and second, and between the third and fourth terms, we infer that the genus must be defined by means of some general *relation* subsisting in common between the first and second, and between the third and fourth, of all proportional numbers. But from inspecting the above examples, it appears that this general relation cannot be expressed by like parts of the first and third, or of the second and fourth terms: for, in the example of the first species, the first term 7 and the third 13, being prime numbers, have no like parts; besides, all the species, except the fourth, admit the unit as a proportional term; and we cannot take a part of a number, and a like part of an impartible unit. We therefore see that a true definition of proportional numbers cannot be formed from the like parts of the first and third, or of the second and fourth terms. Let us next inquire whether the general relation, constituting the proportionality of numbers, can be expressed by means of like multiples of the first and third, or of the second and fourth terms. Here we know that in any four numbers, like multiples of the first and third, or of the second and fourth, may always be taken. Keeping this in view, let us carefully examine the above examples of the four species of proportional numbers. In the first species, the second term 7 is a part of (21) a multiple of the first, and the fourth term 13 is a like part of (39) a like multiple of the third; in the second species, the second term 4 is a part of (24) a multiple of the first, and the fourth term 5 is a like part of (30) a like multiple of the third; in the third species, the second term 18 is a part of (36) a multiple of the first, and the third term 12 is a like part of (24) a like multiple of the third; and in the fourth species, the

second term 12 is a part of (48) a multiple of the first, and the fourth term 6 is a like part of (24) a like multiple of the third. From the general relation here discovered in the four particular examples of the species, we now presume that the following is a true definition of proportional numbers: viz. *Four numbers are proportional, when the second is a part of a multiple of the first, and the fourth is a like part of a like multiple of the third.** Again, we inquire whether this definition, in its application to the four species of proportional numbers, is universally true. This will appear from the demonstrations of the four following propositions.

PROP. I.

If the first of four numbers be equal to the second, and the third be equal to the fourth, these numbers are proportional.

A . . . . B E . . . . . F
K . . . . I M L

C . . . . D G . . . . . H

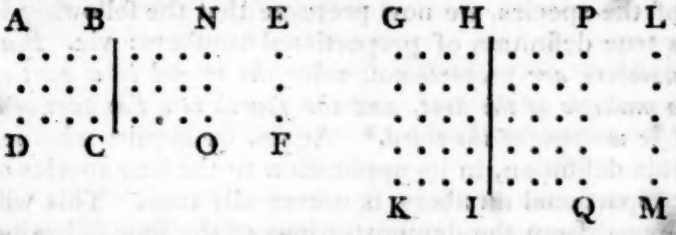
For, let AB, CD, EF, and GH, represent any four numbers, in which AB=CD, and EF=GH; also let ABIK be any multiple of AB, and EFLM a like multiple of EF; then, by inspection, it is evident that CD is a part of (ABIK) a multiple of AB, and that GH is a like part of (EFLM) a like multiple of EF: therefore, by our definition, AB : CD : : EF : GH, as was to be shown.

Corollary. In four proportional numbers, if the first be equal to the second, the third is equal to the fourth.

* The following definition has here been rejected as incompatible with the true idea of ratio; viz. Four numbers are proportional when the first is a part of a multiple of the second, and the third is a like part of a like multiple of the fourth.

PROP. II.

If the first of four numbers be a multiple of the second, and the third be a like multiple of the fourth, these numbers are proportional.

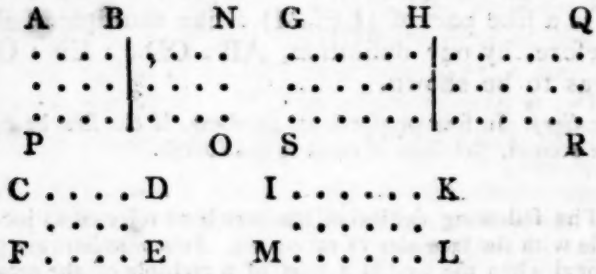


For let ABCD, EF, GHIK, and LM, represent any four numbers, in which ABCD is a multiple of EF, and GHIK a like multiple of LM; also let ANOD be any multiple of ABCD, and GPQK a like multiple of GHIK; then it is evident, by inspection, that EF is a part of (ANOD) a multiple of ABCD, and that LM is a like part of (GPQK) a like multiple of GHIK: hence, by our definition, ABCD : EF : : GHIK : LM, which was to be shown.

Corollary. In four proportional numbers, if the first be a multiple of the second, the third is a like multiple of the fourth.

PROP. III.

If the first of four numbers be a part of the second, and the third be a like part of the fourth, these numbers are proportional.



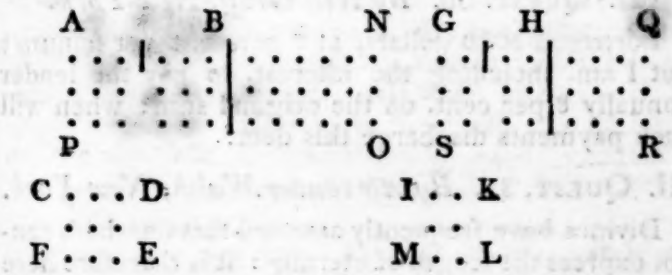
For, let AB, CDEF, GH, and IKLM, represent any

four numbers, in which AB is a part of CDEF, and GH a like part of IKLM; let ANOP be a multiple of AB, and GQRS a like multiple of GH; then, by inspection, CDEF is a part of (ANOP) a multiple of AB, and IKLM is a like part of (GQRS) a like multiple of GH: hence, by our definition, AB : CDEF : : GH : IKLM, as was to be shown.

Corollary. In four proportional numbers, if the first be a part of the second, the third is a like part of the fourth.

PROP. IV.

If the first of four numbers be a multiple of a part of the second, and the third be a like multiple of a like part of the fourth, these numbers are proportional.



For, let AB, CDEF, GH, and IKLM, represent any four numbers, in which AB is a multiple of a part of CDEF, and GH a like multiple of a like part of IKLM; let ANOP be a multiple of AB, and GQRS a like multiple of GH; then, from inspecting the figure, it is plain that CDEF is a part of (ANOP) a multiple of AB, and that IKLM is a like part of (GQRS) a like multiple of GH: therefore, by our definition, AB : CDEF : : GH : IKLM, which was to be shown.

Corollary. In four proportional numbers, if the first be a multiple of *some* part of the second, the third is a like multiple of a like part of the fourth.

Since, therefore, we have found our definition to be universally correct, this inquiry is finished.

ARTICLE XIII.

New Questions to be solved in the next Number.

I. Quest. 29. *By A. Rabbit, Harlaem, near New-York.*

"A father dying, left £500 in cash, with five bills, each £48 10*s.* 6*d.* He ordered £20 to be laid out on his funeral, and his debts to be paid, amounting to £164. The residue he bequeathed to his five sons, as follows: to the eldest $2\frac{2}{5}$ shares, and to each of the other four an equal portion of what remained. How much ought each son to receive?"

II. Quest. 30. *By Niel Gray, New-York.*

Borrowed 5000 dollars, at 7 per cent per annum; but I am, including the interest, to pay the lender annually 8 per cent. on the original sum: when will such payments discharge this debt?

III. Quest. 31. *By Alexander Walsh, New-York.*

Divines have frequently asserted that numbers cannot express the length of eternity: it is therefore here required to show those reverend gentlemen, that the number of years contained in eternal duration may be accurately expressed by means of two digits.

IV. Quest. 32. *By John Smithis, Philadelphia.*

Prove, by algebra, that the arithmetical is greater than the geometrical mean.

V. Quest. 33. *By W. Thompson, Charleston (S.C.)*

Demonstrate the existence of surd quantities.

VI. Quest. 34. *By the Rev. J. Blackburn, Cambridge, England.*

The height of a tower, erected on a horizontal plane, is 118 feet. At the top of this tower, by means

of a piece of board, a line and a plummet, I found that a straight line bisecting the complement of the angle of depression of an object, at a distance on the plane, struck the same plane, 115 feet from a point in the base of the tower, directly under the place where I stood. What distance was the object from the top of the tower?

VII. QUEST. 35. *By W. Lenhart, York-Town, Pennsylvania.*

In a triangle, having given one side, and an angle adjacent to that side, it is required to determine, by a geometrical construction, the other two sides, when the sum of their squares is a minimum.

VIII. QUEST. 36. *By J. D. Craig, Philadelphia.*

My little boy would fondly know,
To strike his hoop that it may go
The swiftest 'long the street:
Pray be so kind, the place to find
The stroke the hoop must meet.

IX. QUEST. 37. *By Diarius Yankee, Bunker's Hill.*

Hereafter the President of the United States and the King of Spain will settle all their national disputes by means of two telegraphs. For this purpose, one tower will be erected at Washington city, in lat. 38° 53′ N. long. 77° 15 W. and another at Madrid, in lat. 40° 25′ N. long. 3° 12′ W. The height of the republican will be to that of the royal tower as 5 to 4. Moreover, a Massachusetts man, stationed on the top of the republican tower, will regularly, at the hour of 2 A. M. by the sound of a conch-shell, proclaim to his Catholic Majesty all necessary negociations. The earth being spherical, I

demand the heights of the towers, and the time of the day at Madrid, when the sound of the shell will be heard on the top of the royal tower.

X. QUEST. 38. *By Robert Adrain, York-Town, Pennsylvania.*

Given the sum of the squares of three numbers= b, the sum of their cubes=c, and the sum of their biquadrates=d; to find the numbers.

XI. PRIZE QUEST. 39. *By G. Baron, New-York.*

[The author of the best solution to this question shall receive a handsome silver medal, value six dollars.]

A Fellow of the American Academy of Arts and Sciences was sent, by the State of Massachusetts, to spy out the land of Louisiana. Arrived in that country, he entered a cavern in one of the mountains, and discovered a solid body of pure gold, in the form of the frustum of a right cone, lying with its slant side upon a horizontal plane. The diameter of its smaller end measured 48 feet; but in attempting to measure that of the larger, an imperceptible touch of his measuring rule tumbled the body on its smaller end: Alarmed by this terrible accident, with a trembling hand he ventured only to measure the diameter of the larger end, which was 60 feet; and after making the best of his way home, he communicated this discovery to the Governor of the Commonwealth, who wisely began to consider the purchase of Louisiana as advantageous to the Union. O ye learned sages of the Southern States, the words expressing the sequel of this story are unlawful to utter! and therefore (for ye know that a pennyweight of pure gold is worth a dollar) I dare but say, that, without your assistance, the immense value of this golden frustum must remain eternally unknown.

ARTICLE XIV.

Solutions of the Questions proposed in Article XIII.

I. Quest. 29. *Solved by A. Rabbit, Harlaem, near New-York.*

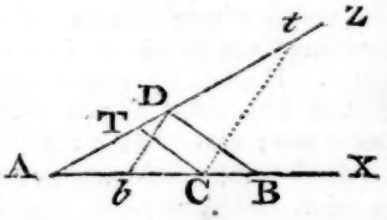
£. 500+5×(£. 48 10s. 6d.)—(£. 20+£. 164)= £. 558 12s. 6d.=the net estate, bequeathed to the five sons; whose shares (if the question have any meaning) are as $2\frac{2}{7}$, 1, 1, 1 and 1, or as 12, 5, 5, 5 and 5. Hence, by proportion 12+4×5=32 : 12 : : £. 558 12s. 6d. : £. 209 9s. $8\frac{1}{4}$d. the share of the eldest son; and 12 : 5 : : £. 209 9s. $8\frac{1}{4}$d. : £. 87 5s. $8\frac{7}{16}$d. the share of each younger son. *N. B.* This question is taken from Shepherd's Columbian Accountant, page 48, where the answers are wrong, and evidently deduced from an erroneous principle. In this country authors of arithmetic have lately sprung up like a parcel of mushrooms, and it would have been well for the young and rising generation had the former been as harmless as the latter. These upstart authors have most perniciously corrupted, distorted, and degraded the noble and useful science of numbers, and metamorphosed our sons into mere counting machines, moving according to a heterogeneous collection of unscientific and stupid rules. A good book of arithmetic is much wanted in America; but so long as the wretched productions of Pike, Walsh, Shepherd, and Co. are encouraged, we cannot expect a man of talents to enrol his name in our list of numerical authors.

II. Quest. 30. *Solved by John Capp, Harrisburg, Pennsylvania.*

Put p=5000, $p\times\cdot08$=400=a, r=·07 and x=time required; then from the question r the rate of in-

terest, a an annuity, and p its present worth are given, to find the time of continuance $x = \frac{\log. a - \log. a - pr}{\log. r+1}$ $= 30{\cdot}734$ years, the time sought. See cor. 2. prob. 34, book 2, Emerson's Algebra.

III. QUEST. 31. *Solved by G. Baron, New-York.*



Let the indefinite straight lines AX and AZ form any angle ZAX; and on AX let AB$=x$ represent a variable and abstract quantity, of which AC is the unit. On AZ, let AD be=the applicate unit, 1 year. Join BD, and through C draw CT parallel to BD. Then, by similar triangles, AB$=x$: AC$=1$: : AD$=1$ year : time AT$=\frac{1}{x}$. Imagine the point B to move towards A; then, during this motion, AB$=x$ will continually decrease; BD will revolve round the point D towards A; CT, parallel to BD, will revolve round the point C towards Z; and the time AT$=\frac{1}{x}$ will continually increase. But when the point B arrives at A, AB$=x$ becomes *nothing*;* DB coincides with DA; CT, parallel to DB, is then parallel to AT; and the time AT becomes *infinite*. Hence the infinite time AT$=\frac{1}{x}$ now$=\frac{1}{0}$, expresses the number of years contained in eternal duration, as was required to be shown.

* See Mathematical Correspondent, p. 62, and the Postscript to the second edition of Emerson's Algebra.

Corollary. Instead of making AD=the applicate unit 1 year, let AD=n=any number of years; then as above $x : 1 :: n$ years : $\frac{n}{x}$ years=AT. But it has been shown, that, when x becomes *nothing*, the time AT is infinite; and therefore if n denote any number of years whatever, $\frac{n}{0}$ will express the number of years in eternity.

Scholium. We must here be careful to remember, that, in the expressions $\frac{1}{0}$ and $\frac{n}{0}$, the numerators 1 and n are years; but the common denominator 0 is the *nothing* of an abstract quantity x.

IV. QUEST. 32. *Solved by John Coope, Chester County, Pennsylvania.*

Put $s=\frac{1}{2}$ the sum of the extremes, and $d=\frac{1}{2}$ their difference; then $s+d$=the greater, and $s-d$=the less extreme: also s=the arithmetical, and $\sqrt{s+d \times s-d} = \sqrt{s^2-d^2}$=the geometrical mean. But s is evidently greater than $\sqrt{s^2-d^2}$, and therefore the arithmetical is greater than the geometrical mean.

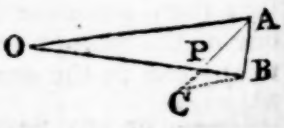
V. QUEST. 3. *Solved by Robert Adrain, York-Town, Pennsylvania.*

Let $\frac{M}{N}$ be any fraction in its least terms; I say, that any power of this fraction is also in its least terms: for let p, q, r and s be the prime factors in the numerator, and a, b, c and d those in the denominator; then, since the fraction is in its least terms, not one of the primes in the denominator can be equal to any one in the numerator; the square, therefore, $\frac{ppqqrrss}{aabbccdd}$, or any power of the fraction, cannot have

any of the prime factors in the numerator and denominator equal one to another; so that the power of the fraction is still a fraction, and cannot become an integer.

Surd quantities are such as cannot be accurately expressed in numbers: of these the most obvious are the inexpressible roots of numbers in various cases, as the square root of 2, &c. In general the roots of integers are either integers or inexpressible in numbers, and consequently surds; the existence of which may be demonstrated in the following manner: Let A be any integer not having an integral root, I say that its root is a surd: for, if possible, let its root be rational, or expressible in numbers. Now, since the root is not an integer, it must consist of an integer, and an irreducible fraction less than unity, and is properly expressed by $a+\frac{m}{n}$; a being the integer, and $\frac{m}{n}$ the fraction, which is in its least terms; then $\frac{an+m}{n}$ is also a fraction in its least terms, and by the principle above shown, any power of this fraction is an irreducible fraction. But by hypothesis a power of $\frac{an+m}{n}$=the integer A; the same quantity, therefore, is both an integer and an irreducible fraction, which is impossible; and hence the root A is a surd quantity. *Q. E. D.*

VI. QUEST. 34. *Solved by the Rev. Thomas P. Irving, Newbern, North-Carolina.*



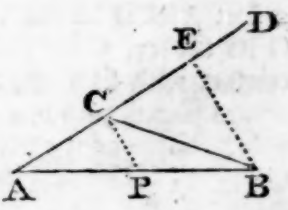
Let OB be the horizontal plane, AB the height of the tower, O the object, and AC the straight line bisecting BAO, the complement of the angle of depression, and intersecting the plane OB in P. In

the right angled triangle ABP, AB=118 and BP=115 feet are given to find the angle BAP=44° 15′ 44″·4: consequently its double 88° 31′ 28″·8=BAO, the complement of the angle of depression. Again, in the right angled triangle ABO, the side AB and the angle BAO are given, to find AO, the distance of the object O from the top of the tower=4583·1 feet.

The same, solved by Robert Adrain, York-Town, Pennsylvania.

Let AB=118 feet=the given height of the tower, O the place of the object, and AP the straight line which bisects BAO, the complement of the angle of depression, making BP=115 feet, on the horizontal plane BO: also let BC, parallel to AO, meet AP produced in C. Because BC is parallel to AO, the angle BCA=OAC=BAC; therefore BC=BA, and by pro. 27, book 2, Emerson's Geometry, CP.PA=$\overline{BA}^2-\overline{BP}^2$. By similar triangles CP : PA : : BC=BA : AO; but CP : PA : : CP.PA : $\overline{PA}^2$: : $\overline{BA}^2-\overline{BP}^2$: $\overline{BA}^2+\overline{BP}^2$,, consequently $\overline{BA}^2-\overline{BP}^2$: $\overline{BA}^2+\overline{BP}^2$: : BA : AO=4583·093 feet, the distance sought.

VII. QUEST. 35. *Constructed by R. Patterson, jun. Philadelphia.*



Let AB be the given side, and DAB the given angle; from the point B draw BE perpendicular to AD, bisect AE in C, and join CB; then is ACB the triangle required. For by prop. 12, book 2, Euclid, $\overline{AB}^2=\overline{BC}^2+\overline{AC}^2$ +2.AC.CE. And since $\overline{AB}^2$ is a constant magnitude, it is evident that $\overline{BC}^2+\overline{AC}^2$ is a minimum

when AC.CE is a maximum. But it is easily inferred from prop. 5, book 2, Euclid, that AC.CE is a maximum, when, agreeable to the above construction, AC=CE: therefore, in the triangle ACB, in which AB and the angle CAB are given, the sum of the squares of AC and CB is a minimum, as was required.

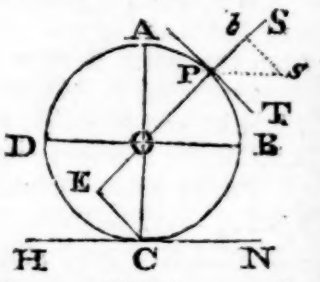
Corollary. When the given angle DAB is either equal to or greater than a right angle, this problem is impossible.

The same, constructed by the Proposer, William Lenhart, York-Town, Pennsylvania.

Let DAB be the given angle, and AB the given side; bisect AB in P, through P draw PC perpendicular to AD, and join CB: then ACB is the triangle required. For CP is drawn from the angle C to the middle of the base AB, and therefore, by prop. 28, book 2, Emerson's Geometry, $2.\overline{AP}^2+2.\overline{CP}^2=\overline{AC}^2+\overline{CB}^2$=a minimum, by the question. But since AP is a constant magnitude, $\overline{AC}^2+\overline{CB}^2$ is a minimum when PC is a minimum; and, according to this construction, PC is a minimum when it is perpendicular to AD, for the angle DAB is given; therefore $\overline{AC}^2+\overline{CB}^2$ is a minimum, as was required.

VIII. QUEST. 36. *Solved by Thomas Maughan, Quebec.*

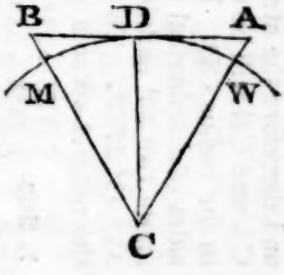
Let ABCD be the hoop, O its centre, C, its point of contact with HN the horizontal plane, and P a variable position of the point in question. The impinging force of the blow at the point P will be the greatest possible when it is struck in the direction of the radius.



For let OPS be a radius

produced through the point P; and in any other direction draw P*s*=PS, the former representing the absolute force of the blow in the direction P*s*, and the latter that in the direction PS; from *s* draw *sb* perpendicular to OS; then *b*P is the impinging force of the blow when struck in the direction *s*P, and SP is the impinging force of the same when struck in the direction SP. Hence PS=P*s*, is evidently greater than *b*P, as was to be shown. Again, it is plain that the impinging force alone must here turn the hoop about C, the centre of motion: for *bs*, parallel to the tangent PT, cannot contribute to that effect. Let us now suppose the hoop to be struck at P, in the direction of the straight line SPO, in which SP represents the absolute force of the blow; on SPO produced, if necessary, let fall the perpendicular CE: then, by mechanics, the effect of the force SP, in turning the hoop about the centre of motion C, is as SP.CE, =a maximum by the question. But since SP is constant, this effect must be a maximum when CE is a maximum; that is, when CE coincides with and becomes equal to CO; or when P coincides with B, and SPE coincides with the direction of BD, the diameter parallel to the plane of the horizon. Hence, then, the hoop must be struck in a horizontal direction, on the end of the diameter which is parallel to the horizon.

IX. QUEST. 37. *Solved by Robert Adrain.*



On the spherical earth whose centre is C, let W and M represent the positions of Washington and Madrid, and MDW the arc of a great circle intercepted between those cities; in CW and CM produced let WA and MB represent the two towers, of such heights that their summits may just be visible from each other,

and therefore the straight line AB will touch the arc WM in some point D; consequently CA and CB will be the secants, and DA and DB the tangents of the arcs DW and DM, to the radius CD. From the given latitudes and longitudes of the places we have two sides of a spherical triangle, viz. the co-latitudes 51° 7′ and 49° 35′, and the contained angle 74° 03′ =the difference of longitude, from which the third side WDM is found= 55° 15′ 37·3″. To find DW and DM we shall make use of trial and error by means of the natural secants to the radius CD=1, as follows:

1. Sup. { DW=29° 00′ 00″ / DM=26° 15′ 37·3 } natural secants —rad. { AW=·1433541 / BM=·1150849 } div. by { 5 / 4 } quot. { =286708 / =287712 } diff. =1004

2. Sup. { DW=29° 1′ 00″ / DM=26° 14′ 37·3 } natural secants —rad. { AW=·1435385 / BM=·1149251 } div. by { 5 / 4 } quot. { =287077 / =287313 } diff. = 236

Diff. of diff. = 768

Now, say as 768 : 236 : : 60″ : 18″·4, which, added to 29° 1′, gives the true value of DW =29° 1′ 18″·4, and DM=26° 14′ 18″·9. Of these arcs DW, DM, the natural secants —radius, are ·1435951 and ·1148760, which are the values of AW and BM to the radius unity; also the tangents DA, DB, are ·5548061 and ·4928977, and their sum AB= 1·0477038. Now let these values of AW, BM, and AB, be multiplied by 21000000= the number of feet in the radius of the earth, and their true values will be as follows: AW=3015497 feet=571 English miles and 617 feet, BM=2412396 feet=456 English miles and 4716 feet; also AB=22001780 feet, which, divided by 1142, quotes 19266 seconds of time=5h. 21m. 6s.=the time of the flight of sound from A to B. The dif-

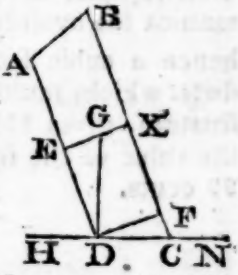
ference of long. 74° 3′ converted into time, gives 4h. 56m. 12s. Therefore 2h.+4h. 56m. 12s.+5h. 21m. 6s.=17*m*. 18s. P. M. the true time of the day at Madrid when the sound will be heard at the top of the royal tower. Q. E. I.

X. QUEST. 38. *Solved by John Capp, Harrisburg, Pennsylvania.*

Let the three numbers sought be denoted by x, y, and z. Put $x+y+z=a$, $xy+xz+yz=p$, and $xyz=q$; then by prob. 75, sect. 11, Simpson's Algebra, $a^2-2p=b$, the sum of the squares of the three required numbers, $a^3-3ap+3q=c$, the sum of their cubes, and $a^4-4a^2p+4qa+2p^2=d$, the sum of their biquadrates. From these three equations we readily obtain $a^4-6ba^2+8ca+3b^2-6d=0$, which, when $b^2=2d$, becomes $a^3-6ba+8c=0$, a cubic; and, consequently, by the rules of biquadratics, or of cubics, according to the case, the value of a will be found: and hence p and q will be determined. Lastly, from the three equations $x+y+z=a$, $xy+xz+yz=p$, and $xyx=q$ (a, p, and q being now given quantities), we have $z^3-az^2+pz-q=0$, the three roots of which are evidently $=x$, y, and z, the numbers required.

XI. PRIZE QUEST. 39. *Solved by John D. Craig, Philadelphia.*

It is well known that the convex surface of a conic frustum touches a plane in a straight line; and that when such a frustum lies with its slant side upon a horizontal plane, a vertical plane passing through the horizontal line of contact will also pass through the axis of the frustum. Now let DC represent the line of contact of our golden frustum with HN,



the plane of the horizon, ABCD a section of this frustum cut by a vertical plane passing through DC, and consequently through EX, the axis of the frustum. Then, since an imperceptible touch tumbled the body on its smaller end, it is evident, from a principle in mechanics, that a straight line DG perpendicular to HN, must intersect the axis EX in G, the centre of gravity of the frustum. Draw DF parallel to EX, and put BC=b=60, AD=a=48, and EX the altitude of the frustum=x; then from the nature of a right cone $FC=\frac{b-a}{2}=6=d$, and by art. 12, Dr. Hutton's Misc. Math. $EG=\frac{3b^2+2ab+a^2}{b^2+ab+a^2}\times\frac{1}{4}x$, the distance of the centre of gravity from the smaller end of the frustum, which expression may be assumed=$\frac{1}{4}mx$. Further in the similar triangles DEG and DFC, $DF=x:FC=d::DE=\frac{1}{2}a:EG=\frac{ad}{2x}=\frac{1}{4}mx$; consequently $x=\sqrt{\frac{2ad}{m}}=24\sqrt{\frac{61}{131}}=16\cdot37723238090048$ feet, the altitude of the frustum; and therefore, by mensuration, the solidity of the frustum will be found =37661·8340276659 feet. Lastly, the specific gravity of common water is to that of pure gold as 1 to 19·64, a cubic foot of water weighs 1000 ounces avoirdupoise, and according to the Encyclopædia Britannica the avoirdupoise pound is=7004 grains troy; hence a cubic foot of pure gold weighs 358225·416 dwts. which, multiplied into the above solidity of the frustum, gives 13491426186·991 dwts. consequently the value of the frustum is 13491426186 dollars and 99 cents.

ACKNOWLEDGEMENTS, &c.

A list of the names and places of residence of the ingenious gentlemen who have given true solutions to the questions proposed in No. IV. The figures annexed to the names refer to the questions answered by each, as numbered in the work.

R. Adrain, York-Town, Penn.	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11.
G. Baron, New-York,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11.
John Capp, Harrisburg, Penn.	1, 2, 4, 10.
John Coope, Chester Co. Penn.	4.
John D. Craig, Philadelphia,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11.
John Craggs, near Richmond, V.	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11.
Rev. T. P. Irving, Newbern,	1, 2, 3, 4, 5, 6, 7, 8, 9, 11.
W. Lenhart, York-Town, Penn.	7.
J. M'Ginness, Middletown, Penn.	1, 2, 3, 4, 6, 7, 9, 11.
Thomas Maughan, Quebec,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11.
James North, Philadelphia,	11.
R. Patterson, jun. Philadelphia,	1, 2, 3, 4, 5, 6, 7, 9, 11.
A. Rabbit, Harlaem, New-York,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11.
J. Romer, Middletown, Penn.	1, 6, 7, 8, 9, 10, 11.
J. Smithis, Philadelphia,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11.
T. Whittaker, Harrisburg, Penn.	1, 2, 4, 6, 11.
James Temple, New-York,	1, 2, 3, 4, 6, 7, 9, 11.
Diarius Yankee, Bunker's Hill,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11.
N. Young, Lexington, Ken.	1, 2, 6, 7, 9, 11.

The Prize-Medal (value six dollars) has been awarded to *John D. Craig*, Philadelphia, who is requested to authorize some person to receive the same from Mr. *George Baron*, New-York. This Medal will be ready for delivery on the first day of June next.

ARTICLE XV.

A Disquisition concerning the Motion of a Ship which is steered on a given Point of the Compass. By ROBERT ADRAIN.

DEFINITION.

In the following essay a ship is said to be *steered* on a given point of the compass when she sails in

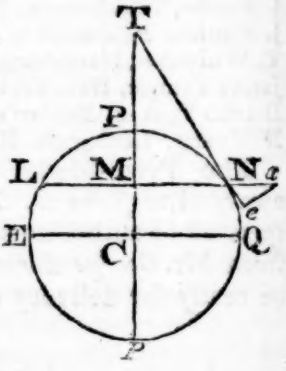
such a manner that her keel or upright longitudinal section coincides with the meridian on which she is, or makes an angle with each successive meridian equal to the angle contained between that meridian on the same side of it towards the same parts and the proposed point of the compass.

Corollary. According to this definition a ship may be steered on one point of the compass and yet sail on a different point.

Proposition I. Problem.

If a tube be laid along a given parallel of latitude, and a ball be projected therein with a given velocity, it is required to determine the pressure of the ball against the tube in a horizontal direction, granting the earth's true figure and rotary motion.

Solution.



Let PQ*p*E be a meridional section of the earth through the axis PC*p*, P and *p* being the poles, C the centre, ECQ a diameter of the equator, and LMN a diameter of the parallel along which the tube is laid. Let the straight line NT touch the meridian in N, and meet the axis of the earth produced in T; also in MN produced take any point *a*, and on TN produced let fall the perpendicular *ae*. The earth being an oblate spheroid, its simple attraction on a particle of matter at any point N between P and Q, is not at right angles to the surface at N, but gives the particle a tendency to move along the meridian towards the nearest pole P. This tendency of the particle at N is precisely balanced and counteracted

by the effect of the centrifugal force of the particle remaining on the same point of the parallel on the earth, and consequently revolving uniformly in the circumference of a circle about the centre M, by the earth's uniform rotation on its axis, in 23 hours 56 minutes and 4 seconds. The particle therefore at N, which continues on the same point of the earth's surface, has no absolute tendency either towards the equator or the nearest pole: were this not the case, it is manifest that the earth could not preserve its present form. That the centrifugal force of a particle at N revolving in the parallel, gives it a tendency to move along the meridian towards the equator, is easily demonstrated thus: The particle at N revolving in the circumference of a circle about M, has a centrifugal force, by which it endeavours to recede, in the plane of that circle, from the centre of its circular motion. Let N*a* be the measure of this centrifugal force, and let the force N*a* be resolved into the two forces N*e*, *ea;* with the force N*e* it is evident the particle or ball tends along the meridian towards the equator; but the force *ea* being at right angles to the earth's surface at N, has no effect on the ball in a horizontal direction. When the ball therefore is relatively at rest on the surface of the earth, that is, when it continues on a given point of the parallel, it is in fact revolving eastwardly with a uniform motion about the centre of the parallel, by the earth's uniform rotation on its axis; and, in consequence of this, it tends along the meridian NQ towards the equator. But this tendency being precisely equal to the effect of gravity on the ball in the opposite direction, the ball in this case has no pressure whatever on the tube in a horizontal direction. If the ball be projected eastwardly in the tube, its absolute velocity will now be greater than before, and its centrifugal force being also greater, its tendency along the meridian towards the equator will exceed the force of

gravity in the opposite direction; and by this excess the ball will press against the tube horizontally towards the equator. If the ball be projected westwardly with a relative velocity in the tube less than twice the absolute velocity of a given point of the parallel eastwardly, the absolute velocity of the ball westwardly or eastwardly round the axis of the earth will be less than the absolute velocity eastwardly of a given point of the parallel, and therefore its centrifugal force will be less than if it had been relatively at rest on the parallel at the given point N; consequently its tendency along the meridian towards the equator will be less than the force of gravity in the opposite direction, so that in this case the ball, by that excess, will press horizontally on the tube towards the nearest pole. When the relative velocity of the ball in the tube westwardly is just equal to the absolute velocity eastwardly of a given point of the parallel, by the earth's rotation about its axis, the ball will have no angular motion round the axis, and therefore no centrifugal force, and consequently no tendency along the meridian towards the equator; its horizontal pressure against the tube towards the nearest pole is therefore in this case a maximum, being the whole effect of the horizontal gravity. When the relative velocity of the ball in the tube westwardly is equal to twice the absolute velocity eastwardly of a given point of the parallel, the absolute velocity of the ball westwardly will be equal to the absolute velocity of a given point of the parallel eastwardly; therefore, in this case, the effects of gravity and the centrifugal force along the meridian being equal to each other, the ball will have no horizontal pressure on the tube. Lastly, the reader will easily see, that when the relative velocity of the ball in the tube westwardly is greater than in the last case, the ball will press horizontally against the tube towards the equator.

Let us now investigate the absolute quantity of the pressure, and compare it with the weight of the ball. Put a=the given relative celerity of the ball in the tube, v=the absolute velocity of a given point of the parallel, by the earth's rotation on its axis eastwardly, x=MN the radius of the parallel, t=NT the tangent of the elliptic polar distance PN, and g=the measure of the accelerative gravity towards the earth at N, which is $32\frac{1}{6}$ feet nearly. The measure of the ball's centrifugal force in the direction Na is well known to be $\frac{v^2}{x}$, when it continues on a given point N of the revolving parallel; and by similar triangles, as TN : MN :: Na : Ne, that is, as $t : x :: \frac{v^2}{x} : \frac{v^2}{t}$, which is the measure of the ball's tendency along the meridian NQ towards the equator, resulting from its centrifugal force when at relative rest in the tube, and therefore is also the true measure of the force of gravity in the opposite direction, arising from the oblate figure of the earth. But when the ball moves eastwardly in the tube, with the velocity a, its absolute velocity eastwardly is $v+a$; consequently its centrifugal force Na is $\frac{\overline{v+a}^2}{x}$, and by similar triangles as before, we have Ne=$\frac{\overline{v+a}^2}{t}$=the measure of the ball's horizontal tendency towards the equator, arising from its centrifugal force. From this quantity we must subtract $\frac{v^2}{t}$, which measures the effect of gravity in the opposite direction, and there remains $\frac{\overline{v+a}^2-v^2}{t}=\frac{2av+a^2}{t}$ for the true measure of the ball's horizontal pressure on the tube towards the equator; therefore, in this case, as $g : \frac{2av+a^2}{t}$, so is the weight of the ball to its horizontal pressure on the tube. When the ball has a

relative velocity in the tube westwardly less than v, then $v-a$ is its absolute velocity eastwardly, and $\frac{\overline{v-a}^2}{t}$ its tendency towards the equator in a horizontal direction, arising from its centrifugal force, which being taken from $\frac{v^2}{t}$ leaves $\frac{2av-a^2}{t}$ for the measure of the ball's horizontal pressure against the tube towards the nearest pole. In a similar manner the other cases of this proposition may be readily investigated. Q. E. I.

Cor. 1. In computing the pressure, it will be sufficiently accurate to consider t as the tangent of the co-latitude, to r the radius of the earth taken as a globe.

Cor. 2. By substituting for v its value in termsof x, r, and b the velocity of a given point of the equator revolving about the earth's axis, we shall obtain other formulas exhibiting the quantity of pressure sought.

Cor. 3. If any body whatever move along a parallel of latitude eastwardly or westwardly with any proposed velocity, it is manifest that its absolute tendency to or from the equator in a horizontal direction is truly measured by the preceding formulas, according to the several cases.

Lemma. If a ship be steered on a given point of the compass, she must necessarily sail on the same point when the forces which act on her towards opposite sides, at right angles to her length, such as hydrostatic pressure, centrifugal force, gravity, &c. exactly balance each other, and mutually destroy each other's effects; but if those opposite forces be unequal, the ship must gradually leave the line of direction in which she is steered (being acted on by an unbalanced force), and deviate on that side towards which the unbalanced force is directed. This lemma is too evident to require demonstration.

PROPOSITION II. THEOREM.

If a ship in any latitude steers due east with any velocity, or steers due west with a velocity greater than twice the velocity of a given point of that latitude eastwardly, by the earth's rotation on its axis, she will gradually deviate from that latitude, and approach towards the equator: if her velocity due west be equal to twice the velocity of a given point in that latitude moving eastwardly by the rotation of the earth, she will continue in that latitude, and describe the parallel; but if her velocity due west be less than in the case last mentioned, she will gradually deviate from that latitude towards the nearest pole; and when her velocity west is just equal to the rotary velocity east of a point on that latitude, her deviation towards the nearest pole will, *ceteris paribus*, be a maximum.

DEMONSTRATION.

Case 1. Let the ship be steered due east, and suppose, if possible, that she describes the parallel of latitude. The action of the wind may be considered as in a due east direction, because the effect of its obliquity in producing leeway is no part of the present inquiry. The resistance of the water is in a due west direction, and its hydrostatic pressures on the opposite sides of the ship are precisely equal to each other. These things, therefore, have no influence in urging the ship from the parallel. But by *Cor.* 3, *Prop.* 1, she has an unbalanced tendency towards the equator, of which the measure is $\frac{2av+a^2}{t}$, according to the notation made use of in that proposition: therefore, by the preceding lemma, the ship will leave the parallel, and deviate towards the equator; but, by hypothesis, she must continue on the parallel at the same time, which is impossible: and still more unreasonable, if possible, is it to suppose that she would, in this case, deviate from the parallel towards

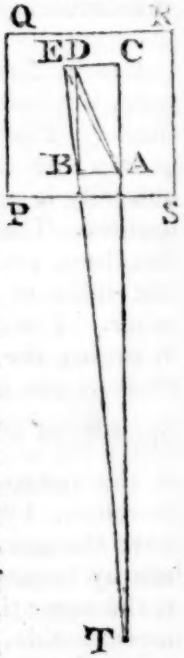
the nearest pole. In the same manner may the remaining cases of this proposition be easily demonstrated.

PROPOSITION III. PROBLEM.

If a tube be laid along a meridian, and a ball be projected therein with a given velocity, it is required to determine the pressure of the ball against the tube in a horizontal direction, granting the earth's true figure and rotary motion.

SOLUTION.

Case 1. Suppose the ball to move *from* the nearest pole. Let PQRS be an exceedingly small part of the earth's surface, which may be considered as a plane. Let the ball be at A, proceeding along the tube towards C, and in the given particle of time in which the ball would move uniformly from A to C, with the velocity at A, let the tube AC, by the earth's uniform rotation eastwardly about its axis, come into the position BE, its points A and C having described the very small unequal arcs of parallels AB and CE. Let CA, EB be continued out in straight lines to meet the earth's axis produced in T, and let BD be drawn parallel to AC. Now AC, AB, and the diagonal AD, measure respectively the velocity of the ball along the tube, its circulatory velocity in the parallel at A about the earth's axis, and its absolute velocity at A. The ball, then at the end of the given time, is found at E, in the parallel CE; but if it had been acted on by no horizontal force, it would have been at D, having described the diagonal AD instead of the curve AE. The pressure of the tube eastwardly has caused the ball in the given time to advance through the space DE at



right angles to the tube; therefore 2DE is the proper measure of the ball's pressure against the tube in a due west direction; and we may add, though it is very obvious that this measure cannot in the least be affected by any gravity, or centrifugal force, which acts in the direction of the tube itself AC.

To compute DE, we have, by similar triangles, as TA : AB : : BD : DE, that is, using the same notation as in the preceding propositions, as $t : v :: a : \frac{av}{t} =$ DE, and $\frac{2av}{t}$ is the true measure of the ball's pressure on the tube in a due west direction; consequently as $g : \frac{2av}{t}$: : the weight of the ball : its horizontal pressure on the tube.

Case 2. By the same method of investigation we discover, that when the ball moves *towards* the nearest pole, its pressure against the tube in a due east direction will be properly expressed by $\frac{2av}{t}$. Q. E. I.

Cor. 1. If any body whatever move along a meridian, from or towards the nearest pole, it will have an absolute horizontal tendency in a due west or due east direction, of which the measure is $\frac{2av}{t}$.

Cor. 2. Because the effect of the ball's centrifugal force, in urging it towards the equator, is equal to that of the horizontal gravity in the opposite direction, it is evident that the ball will proceed either way along the tube with a uniform velocity.

Proposition IV. Theorem.

If a ship be steered in a meridional direction she will not describe a meridian, but when moving from the nearest pole will deviate westwardly, and when moving towards the nearest pole will deviate eastwardly from each successive meridian.

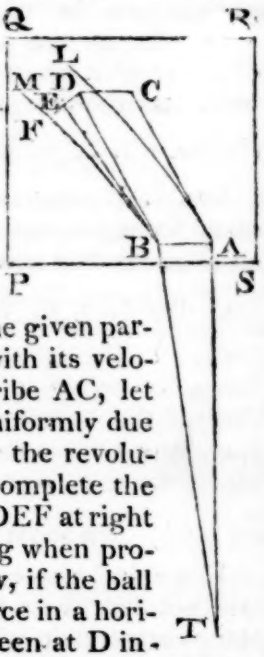
DEMONSTRATION.

Suppose the ship, if possible, to describe a meridian. By *Cor.* 1, *Prop.* 3, she has an unbalanced tendency at right angles to her length, and, therefore, by the *Lemma* she must leave the meridian; but by hypothesis she must continue on the meridian at the same time, which is impossible. Wherefore, &c. *Q. E. D.*

PROPOSITION V. PROB.

If a tube be laid along any given oblique rhumb-line, and a ball be projected therein with a given velocity, it is required to determine the pressure of the ball against the tube in a horizontal direction, granting the earth's true figure and rotary motion.

Solution. Let PQRS be an exceedingly small part of the earth's surface, which may be considered as a plane, and AL a very small part of the tube.



Case 1. Suppose the ball to move in the tube from the nearest pole eastwardly, that is, along the tube AL, from A towards L. Let AC be a tangent to the rhumb-line AL at A, and AT a tangent to the meridian at A, meeting the earth's axis produced in T. In the given particle of time in which the ball, with its velocity at A, would uniformly describe AC, let the whole figure TACL move uniformly due east into the position TBEM, by the revolution of the earth about its axis; complete the parallelogram BACD, and draw DEF at right angles to the tangent BE, meeting when produced the rhumb-line in F. Now, if the ball had not been acted on by any force in a horizontal direction, it would have been at D instead of being at F, at the end of the given time. Excluding therefore the effect of the ball's centrifu-

gal force and the horizontal attraction, the pressure of the tube against the ball eastwardly, and consequently of the ball against the tube westwardly, will be accurately measured by 2DF; the ball's velocity in the tube being measured by AC, and its circulatory velocity about the axis of the earth by AB. And since the earth's attraction on the ball in the direction AT, is precisely counterbalanced by the effect of the centrifugal force in the opposite direction, it follows, that these united have no influence on the horizontal pressure of the ball, or on its motion along the tube, which will therefore continue uniform. We are now to investigate the value of DF, which, in this case, consists of the two parts DE, EF. The part DE is the very same as if the ball had moved along the meridian with the given velocity, for it is easy to see that the triangles TAB and BDE are similar; consequently TA : AB : : BD : DE, that is, $t : v :: a : \frac{av}{t} =$ DE. Again, the curve line BFM being a part of a rhumb-line on the surface of the earth, makes equal angles with all the meridians intersecting it: and since all these meridians within the space PQRS may be taken for straight lines, which continued would pass through T; therefore BFM, on the plane PQRS, is the arc of an equiangular spiral, having its centre in T. Let s=the sine of the given angle TAC or TBM to radius 1, and (still retaining the former notation) the radius of curvature at B being $\frac{t}{s}$, we have $2EF = \overline{BE}^2 \div \frac{t}{s} = \frac{sa^2}{t}$, to which add $2DE = \frac{2av}{t}$, and the sum $\frac{2va+sa^2}{t}$ is, in this case, the measure of the pressure required; therefore as g is to $\frac{2av+sa^2}{t}$, so is the weight of the ball to its horizontal pressure against the tube westwardly.

Case 2. If the ball move from the nearest pole westwardly, the horizontal pressure will be found in a similar manner to be $\frac{2av-sa^2}{t}$ westwardly, supposing $2av$ greater than sa^2; but if less it will be $\frac{sa^2-2av}{t}$ eastwardly.

Case 3. If the ball move towards the nearest pole eastwardly, the pressure sought will be $\frac{2av+sa^2}{t}$ eastwardly.

Case 4. If the ball move towards the nearest pole westwardly, the pressure will be $\frac{2av-sa^2}{t}$ eastwardly if $2av$ exceed sa^2, otherwise $\frac{sa^2-2av}{t}$ westwardly.

Cor. If any body whatever move along an oblique rhumb-line, its horizontal tendency towards the east or west at right angles to its path is truly exhibited in this proposition according to the several cases.

Proposition VI. Treorem.

If a ship be steered on any oblique given point of the compass, she will not describe the true rhumb-line which the compass points out, but will deviate from it towards the east or west, according to the several cases of horizontal pressure demonstrated in the last proposition.

The demonstration of this proposition is derived from the lemma and the corollary to proposition 5, in the very same manner as the demonstration of proposition 4, from the lemma, and corollary 1, proposition 3.

Scholium.

By help of the principles contained in this disquisition, and the known properties of air, we may account in a rational manner for the principal phenomena of the Trade Winds and Monsoons.

ARTICLE XVI.

New Questions to be solved in the next Number.

I. Quest. 40. *By G. Baron, New-York.*

Without any regard to the derivation of words, I wish to know what our mathematicians understand by a *fraction;* and whether the terms of a fraction ought always to be considered as homogeneous quantities.

II. Quest. 41. *By R. Patterson, jun. Philadelphia.*

Given any two sides of a right-angled plane triangle, to discover a general rule, for finding the third side, by a *numerical* operation that shall be the shortest possible.

III. Quest. 42. *By T. Bulmer, Sunderland, England.*

Prove that any number divided by 9, and the sum of its digits divided by 9, leave equal remainders; and determine whether this is an absolute property of the number 9, or merely an effect of our present notation.

IV. Quest. 43. *By J. Craggs, near Richmond, Virg.*

Prove that the sum of the series 1, 3, 5, 7, 9, &c. continued to any number of terms n, is equal to n^2.

V. Quest. 44. *By Diarius Yankee, Bunker's-Hill.*

If 2 be 3, and 3 be 5, and $6\frac{1}{2}$ be 11,
What is the $\frac{1}{2}$ of 26, and the $\frac{1}{4}$ of 27?

VI. Quest. 45. *By Eben. R. White, Danbury, Connect.*

Given one side of a trapezium and the two adjacent angles to find the length of two equal sides, contiguous to the given side, when the area of the trapezium is a given quantity.

VII. Quest. 46. *By W. Lenhart, York-Town, Penn.*

In a plane rectilineal triangle, if from one of the angles adjacent to the base a straight line be drawn perpendicular to the opposite side, then shall the

rectangle contained by the side on which the perpendicular falls, and the part intercepted between the vertical angle and the perpendicular, be equal to the difference of the squares of the straight line drawn from the vertical angle to the middle of the base and the semi-base; required a demonstration.

VIII. QUEST. 47. *By James North, Philadelphia.*

A hole was made in a vessel filled with wine, and, at the same instant, a supply of water was contrived to keep the vessel constantly full: the content of this vessel was 100 gallons, and the hole discharged one gallon of the mixture every 10 seconds. Now supposing the liquor to be always equally diffused throughout the vessel, how much wine remained in the same one hour after the hole was made?

IX. QUEST. 48. *By R. Patterson, jun. Philadelphia.*

The plantations of two neighbours are separated by a zig-zag fence, and it is required to determine geometrically the position of a straight-lined fence, that shall be the shortest possible, to separate these plantations, without altering their respective areas.

X. QUEST. 49. *By Ben. Cheetham, Billiard-Hall.*

Required a general theorem for finding the centre of gravity of the frustum of any pyramidical body.

XI. QUEST. 50. *By the Editors.*

Given $x+y+z=a$, $xy+xz+yz=b$ and $xyz=c$, to find x, y, and z.

XII. PRIZE QUEST. 51. *By Thomas Maughan, Quebec.*

[The author of the best solution to this question shall receive a handsome silver medal, value six dollars.]

A ship from a port in lat. 80° N. and long. 60° W. was steered eastwardly in such a manner that her distance sailed was always equal to her difference of longitude; the earth being spherical, in what latitude would her difference of longitude be 180°, and in what longitude would her latitude be 40° N.?

ARTICLE XVII.

Solutions of the Questions proposed in Article XVI.

I. Quest. 40. *Solved by G. Baron, New-York.*

The question is, "What our mathematicians understand by a *fraction*, and whether the terms of a fraction ought always to be considered as homogeneous quantities?" Were we to examine the various modern treatises written on the subject, we might, probably, conclude that our mathematicians have no determinate idea of a fraction. Some have defined a fraction to be *a part of a number*;* but the fallacy of this definition may be readily shown: for a number is a determinate multitude, or a definite plurality of units, considered without any regard to extension, and that which is not extended cannot be parted or divided; consequently a unit is the smallest possible part of any number. But it is well known to mathematicians, that whatever a fraction may be, it is frequently less than a unit; and hence the definition is false. It may here be observed, that a number has an infinite variety of multiples, but its parts are finite, and often exceedingly limited: thus in the number 12, its multiples are 24, 36, 48, 60, 72, &c. without end; but its parts are limited to 1, 2, 3, 4, and 6, whose denominators are 12, 6, 4, 3, and 2. A prime number has no part but a unit, and the number itself denominates the part. It is, therefore, abundantly evident that the limited nature of the parts of numbers is utterly inconsistent with the general sense in which fractions are universally used. Others have defined a fraction to be *a part of something considered*

* Definitions of the words *multiple*, *part*, and *denominator*, may be seen in page 1 and 2 of this work.

as a whole. Something considered as a whole is a vague expression, to which we cannot easily annex a mathematical idea; but since, from the general usage of fractions, this something is capable of being infinitely divided into parts, it must possess extension, a property which belongs to the subject of geometry, or that of the science of magnitude; for we have already seen, that the division into parts here supposed cannot be universally applied in dividing or separating numbers into parts or equal parcels. Again, from the general usage of fractions we know that this *something*, considered as a whole, and its various parts, are always expressed by means of numbers; but a magnitude and its various parts cannot be universally expressed in numbers, but by means of some known measure, considered as an infinitely divisible unit; and it is well known to mathematicians, that magnitude so expressed is called quantity. Hence then this *something*, considered as a *whole*, must be *quantity*. From these cursory cognitions it appears to me that *quantity*, and not *number*, is the subject of fractions, and that *a fraction is a part of a quantity*. Let now m represent a quantity of any kind, and n any abstract number, then, according to a general custom, $\frac{m}{n}$ will represent the nth part of m, and is, therefore, a fraction. The pure number n, and the quantity m, are the terms of the fraction, and of consequence can never be homogeneous. I am well aware that, to superficial mathematicians, the conclusion here drawn will appear to contradict the scholium in page 95 of this work; but if such men ever discover what is really meant by dividing one fraction by another, the apparent contradiction will instantly vanish.

II. QUEST. 41. *Solved by the Rev. Thomas P. Irving, Newbern, North-Carolina.*

Put h=the hypotenuse, p=the perpendicular, and b=the base of a right angled plane triangle; then from prop. 47, book 1, Euc. we deduce the three following theorems:

$$h=\sqrt{p^2+b^2}=p\sqrt{1+b^2\div p^2}=b\sqrt{p^2\div b^2+1}.$$
$$p=\sqrt{h^2-b^2}=\sqrt{\overline{h+b}\times\overline{h-b}}.$$
$$b=\sqrt{h^2-p^2}=\sqrt{\overline{h+p}\times\overline{h-p}}.$$

Here it is plain that the values of p and b may be readily found by logarithms; and our friend, Mr. White, has, in page 70 and 71 Math. CORT. shown us an easy logarithmic process for finding the value of h. Hence then it appears from these theorems, that the required side may be found, 1st. by extracting the square root of the sum or difference of the squares of the other two sides; 2d. by a table of squares and square roots; 3d. by a table of difference of latitude and departure; and, 4th. by a table of logarithms. No one of these methods will, however, resolve the question by means of an operation, in all cases, the shortest possible; nor does it appear to me that the nature of the subject will afford the required theorem.

The same solved by the Proposer, R. Patterson, jun. Philadelphia.

Put h=the hypotenuse, p=the perpendicular, and b=the base of any right angled plane triangle. Assume $x=h-p$, or $h-b$, according as p is greater or less than b; then, from prop. 47, book 1, Euclid, we have the three following equations:

1. $h^2=\overline{h-x}^2+b^2$ or $\overline{h-x}^2+p^2$.
2. $p^2=h^2-\overline{h-x}^2$.
3. $b^2=h^2-\overline{h-x}^2$.

From the first of these equations, $x=\frac{b^2}{2h-x}$, or $\frac{p^2}{2h-x}$, which, by writing $p+x$ or $b+x$ for h, becomes $x=\frac{b^2}{2p+x}$ or $\frac{p^2}{2b+x}$; from the second, $x=\frac{p^2}{2h-x}$; and, from the third, $x=\frac{b^2}{2h-x}$. Hence we obtain the two following rules.

Rule 1. Divide the square of the less side by twice the greater, always increasing the divisor by the two last figures of the quotient as they occur, and the quotient added to the greater side will give the hypotenuse.

Rule 2. Divide the square of the given side by twice the hypotenuse, always diminishing the divisor by the two last figures of the quotient as they occur, and the quotient, subtracted from the hypotenuse, will leave the side required.

EXAMPLE IN RULE 1.

Given the base and perpendicular of a right angled plane triangle, =30 and 319 respectively, to find the hypotenuse, true to the nearest figure in the sixth place of decimals.

OPERATION.

319=greater side

638)	900	(1·407553
1	639	320·407553=hypotenuse required
639	26100	
14	25616	
6404	48400	
4	44856	
6408	3544	
	3204	
	340	
	320	
	20	

EXAMPLE IN RULE 2.

In a right angled plane triangle the hypotenuse=427, and one of the sides=40, are given; to find the other side, true to the nearest figure of the sixth place of decimals.

OPERATION.

427=hypotenuse

854)	1600	(	1·877665
	1		425·122335=side required
	853		
853	74700		
	18	68096	
8512	660400		
	87	595231	
85033	65169		
	77	59518	
8502,53	5651		
	7	5101	
850,246	550		
	510		
	40		

Scholium by the Editors.

Although we admit that in some particular instances the above two rules may not find the required side by the shortest operation possible; yet, in many of the cases which commonly occur, they will certainly render the calculation shorter than the customary methods. We think proper here to inform our young readers, that in prob. 88, book 1, Emerson's Algebra, these rules are demonstrated and applied to the resolution of quadratic equations.

III. QUEST. 42. *Solved by Robert Adrain.*

Let $r=10$, and a, b, c, &c. the digits of any number in our notation: then the number itself will be ar^2+br+c, from which taking the sum of its digits, $a+b+c$, there remains $a\times\overline{r^2-1}+b\times\overline{r-1}=(a\times\overline{r+1}+b)\times\overline{r-1}=(a\times\overline{r+1}+b)\times 9$: since, therefore, the difference between any number and the sum of its digits is divisible by 9, it evidently follows that the number and the sum of its digits, divided by 9, leave either none or equal remainders. Further, since $r-1$ in any other system would not be equal to 9, this is not a property of the number nine, but merely an effect of our present notation.

Scholium by the Editors.

Our young mathematical friends will easily demonstrate, that the difference of the like powers of any two numbers is divisible by the difference of the same two numbers; or that x^n-y^n is divisible by $x-y$; and consequently that r^n-1 is divisible by $r-1$; and hence they will easily perceive that Mr. Adrain's elegant demonstration is not here limited to any particular number of digits.

The ingenious proposer, Thomas Bulmer, of Sunderland, in England, after answering this question nearly in the same manner as above, adds, "What ought we to think of some of our most eminent English mathematicians of the present day, who have gravely informed us, that this property of the digit 9 was unknown to the ancients? The ancients could not possibly be acquainted with an *effect* before its *cause* existed; and hence our profound sages might just as well have told us that old Pythagoras was never on board an American gun-boat."

IV. QUEST. 43. *Solved by James M'Cormic, Carlisle, Pennsylvania.*

1, 3, 5, 7, 9, &c. continued to n terms, is evidently a series in arithmetical progression, the common difference being 2, and the last term $1+\overline{n-1}\times 2$ or $2n-1$; hence, by a well known theorem, the sum of this series is $=\frac{2n-1+1}{2}\times n=n^2$, as was to be proved.

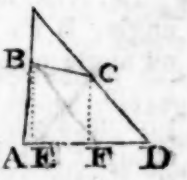
V. QUEST. 44. *Solved by the Proposer, Diarius Yankee, Bunker's Hill.*

By compound proportion $\left\{\begin{matrix}2 & : & 3\\ 3 & : & 5\\ 6\frac{1}{2} & : & 11\end{matrix}\right\}$ :: 13 : 55;

and per quest. $13 : 55 :: \left\{\begin{matrix}\frac{26}{2}\\ \frac{27}{4}\end{matrix}\right\} : \left\{\begin{matrix}55\\ 28\frac{2}{5}\frac{9}{2}\end{matrix}\right\}$ The answers required.

VI. QUEST. 45. *Solved by Robert Adrain.*

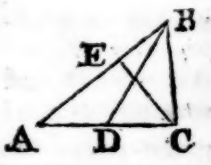
Let d=AD the given side of the trapezium, a= the given area, m=sine, and n=cosine of the given angle A, and p=sine, and q=cosine of the other given angle D: the radius being here considered=1. Also put the length of the required side AB=CD=x; and let BE and CF be perpendicular to AD: then BE=mx, AE=nx, CF=px, and DF=qx; and, consequently, AF=$d-qx$, and DE=$d-nx$. By mensuration BE.AF+CF.DE=twice the given area; that is, $mx\times\overline{d-qx}+px\times\overline{d-nx}=\overline{dm+dp}\times x-\overline{mq+np}\times x^2=2a$, a quadratic equation from which x is easily found.



Scholium by G. Baron.

Some *negative Mathematicians* will probably say that our learned friend ought here to have remarked, that the cosine of an obtuse angle is *negative*. Pray, scientific gentlemen, can any of you give a correct definition of a *negative* quantity?* If such a quantity cannot be defined, it must be either a creature of your own imaginations, or something which you do not rightly understand; and if you do not know what is meant by a *negative* quantity, you can have no definite idea of a *negative* cosine. Let us dive into the ocean of negative mysteries, when we have sounded its depths with the line of philosophy. If you contemplate the above solution, you will easily perceive that the perpendiculars BE and CF must always fall either on AD or AD produced; and that the theorem for finding the double area of the trapezium is universally true. Now, if to each of the different cases which the problem affords, you apply the same reasoning here used by Mr. Adrain, you will be enabled, in a rational manner, to account for all the changes of the signs which will occur in certain terms of the final equation; and hence you will see that this solution is universally true, and not limited to the case of acute angles. Remember that a *solution* and its *final equation* are two things which differ essentially.

VII. QUEST. 46. *Solved by Henry Smith, New-York.*



Let ABC be a plane rectilineal triangle, B its vertical angle, and C an angle adjacent to the base AC. From C, at right angles to AB produced when necessary, draw the straight line CE; and from B to D, the middle point of the base, draw the straight line

* Jared Mansfield, in his Mathematical Essays, has attempted to define a negative quantity. Jared's discourse on this subject reminds me of a Connecticut carle, who said that the moon was exactly the size of a johnny-cake.

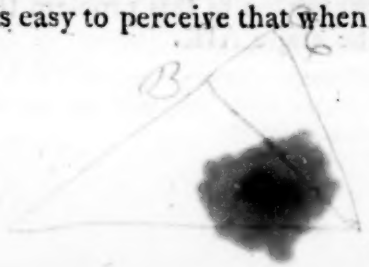
BD. Then by prop. 23 and 22, book 2, Emerson's Geometry, $\overline{AB}^2+\overline{BC}^2=\overline{AC}^2\pm 2.AB.BE=4.\overline{DC}^2\pm 2.AB.BE$. But by prop. 28, book 2, Emerson's Geometry, $\overline{AB}^2+\overline{BC}^2=2.\overline{DC}^2+2.\overline{BD}^2$; consequently $4.\overline{DC}^2\pm 2.AB.BE=2.\overline{DC}^2+2.\overline{BD}^2$, or $\overline{DC}^2\pm AB.BE=\overline{BD}^2$: whence it is evident that $AB.BE=\overline{BD}^2\sim\overline{DC}^2$, as was to be proved.

Corollary. From the above demonstration it is plain that BD is greater or less than DC, according as the angle B is acute or obtuse.

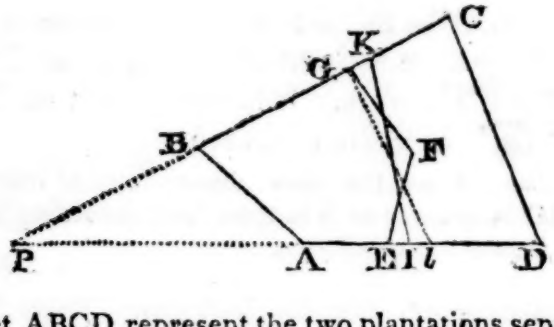
VIII. QUEST. 47. *Solved by the Proposer, James North, Philadelphia.*

Put $c=100$ gallons, the content of the vessel, $s=10$ seconds of time, and $x=$the quantity of wine remaining in the vessel at the end of any variable time t; then from the nature of the question $s:\dot{t}::1:\frac{\dot{t}}{s}=$the quantity of the mixture discharged in the infinitely small particle of time $\dot{t}$; and $c:x::\frac{\dot{t}}{s}:\dot{x}$: whence $\frac{\dot{t}}{cs}=\frac{\dot{x}}{x}$, whose fluent corrected gives hyp. log. $x=$hyp. log. $c-\frac{t}{cs}$. Now, assume $t=3600$ the number of seconds in one hour, and put $n=\cdot 43429448$; and we have com. log. $x=$com. log. $c-\frac{nt}{cs}=\cdot 4365399$; consequently $x=2\cdot 73237$ gallons, the quantity of wine required.

Corollary. The wine would never be entirely exhausted in the vessel; for in the above general equation it is easy to perceive that when $x=o$, t is infinite.



IX. QUEST. 48. *Solved by Ben. Cheetham, Billiard-Hall.*



Let ABCD represent the two plantations separated by the crooked fence EFG, and let BC and AD be straight lines. By a well known problem a straight line G*l* may be drawn, which will separate these enclosures, without varying their respective areas. Then, if BC be parallel to AD, a straight line passing through the middle of G*l*, and meeting BC and AD at right angles, will evidently be the required fence. But when BC and AD are not parallel, they may be produced to meet in some point P. In this case find PI=a mean proportional between P*l* and PG; make PK=PI, and join KI, which will be the fence required. For in the triangles P*l*G and PIK, the angle P is common, and by construction P*l*.PG=$\overline{PI}^2$= PI.PK, or P*l* : PI : : PK : PG; therefore, by prop. 15, book 6, Euclid, the triangles P*l*G and PIK have equal areas;, and consequently KI separates the two plantations, without changing their respective areas. Also, since PI=PK, or the angle PIK=PKI; by the lemma to prob. 68, Appendix to Simpson's Algebra, the straight line KI is a minimum, as was required.

Corollary. In every case of this problem the angle BKI is equal to the angle AIK.

X. QUEST. 49. *Solved by Article* 12, *Dr. Hutton's Miscellanea Mathematica.*

Let ABCD be the frustum of a right cone, EF its axis, and GH a variable straight line intersecting the axis EF, and parallel to DC the diameter of the base of the frustum. Put DC=a, AB=b, AB−DC=d, EF=h, ·785398 &c.=c, and EI=x. Then $h:d::x:\frac{dx}{h}$, and $\frac{dx}{h}+a=\frac{dx+ah}{h}$=GH; hence $c\times\overline{\frac{dx+ah}{h}}^{2}$=the area of the section GH; which drawn into $\dot{x}\times x$, the product is $cx\dot{x}\times\frac{d^2x^2+2adhx+a^2h^2}{h^2}$, whose fluent, when $x=h$ is $ch^2\times\frac{3d^2+8ad+6a^2}{12}$. Lastly, this fluent divided by $ch\times\frac{d^2+3ab}{3}$, the solidity of the frustum, gives $\frac{3d^2+8ad+6a^2}{d^2+3ab}\times\frac{1}{4}h$=(by exterminating d) $\frac{3b^2+2ab+a^2}{b^2+ab+a^2}\times\frac{1}{4}h$=the distance of the centre of gravity from the smaller end of the frustum. And the same theorem will serve for the frustum of any right pyramid, by putting a and b for any like dimensions, in the two ends of the frustum.

Corollary 1. When $a=b$, the frustum becomes a cylinder, and the distance of the centre of gravity from either end=$\frac{1}{2}h$.

Corollary 2. When $a=0$, the frustum becomes a cone, and the distance of the centre of gravity from the vertex=$\frac{3}{4}h$.

XI. QUEST. 50. *Solved by John D. Craig, Philadelphia.*

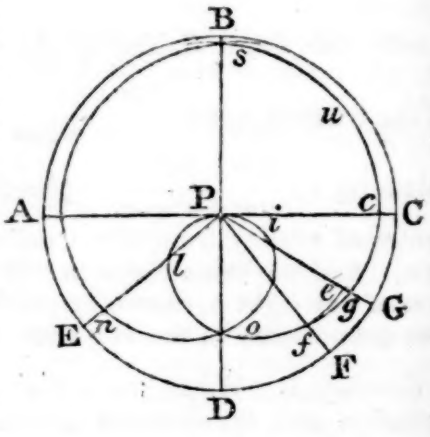
The first equation gives $x+y=a-z$,

The second gives $\overline{x+y}\times z=b-xy$,

And the third gives $xy=\frac{c}{z}$.

From the first of these equations, multiplied by z, and compared with the second, we have $\overline{a-z}\times z=b-xy=b-\frac{c}{z}$, by the third: hence $z^3-az^2+bz=c$. Lastly, since in each of the original equations the three unknown quantities are equally concerned; therefore the three roots of this last cubic equation will be the values of x, y, and z.

XII. PRIZE QUEST. 51. *Solved by John Smithis, Philadelphia.*



Let the annexed figure be the stereographic projection of the convex northern hemisphere of the earth, on the plane of the equator ABCD; and let P represent the north pole, PA, PB, PC, and PD meridians meeting the equator in the points A, B, C, and D. Imagine any meridian PA to revolve uniformly about the pole P towards D, while a point moves from P along PA, so as to describe the curve P*locus*, such that P*lof*, any length of the curve, may be always equal to AEDF, the cor-

responding difference of longitude; then, from the question it is plain that the ship must describe a curve of this nature on the surface of the sea. Infinitely near a coincidence with the meridian PF draw another meridian PG, intersecting the curve *Plofc* in the point *g*, and through the point *f* draw the infinitely small arc *fe* parallel to the equatorial arc FG. Put r=the radius of the earth, $Plof$=AEDF=z, P*f* the complement of the latitude of *f*=L, sine of L=y, and the sine of the course=the sine of the angle Ffg=sine of the angle fge=x, to the same radius r; then FG=$fg=\dot{z}$, $ge=\dot{L}$, cosine of L=$\sqrt{r^2-y^2}$, and the sine of the angle efg=cosine of the course=$\sqrt{r^2-x^2}$. In the infinitely small triangle *fge*, right angled at *e*, $r : x :: \dot{z} : \frac{\dot{x}z}{r}=fe$; and by similar spherical sectors $y : r :: \frac{\dot{x}z}{r} : \frac{\dot{x}z}{y}$=FG=$\dot{z}$: hence $x=y$; and since the sines x and y have the common radius r, the course is constantly equal to the complement of the latitude. Again, in the right angled triangle *feg*, $r : \sqrt{r^2-x^2}=\sqrt{r^2-y^2} :: \dot{z} : \frac{\dot{z}\sqrt{r^2-y^2}}{r}=ge=\dot{L}$; therefore $\dot{z}=\frac{r\dot{L}}{\sqrt{r^2-y^2}}$. Now let L be considered as equal to an arc of latitude reckoned from the equator; then by example 11, prop. 13, sect. 1, of the third edition of Emerson's Fluxions, the fluent z=the meridional parts of L, which needs no correction; for when $z=0$, $\dot{L}$ and the meridional parts of L are each=0. Hence, when a ship is steered according to the hypothesis of this question, the difference of longitude, or the distance of the vessel from the nearest pole, measured on the curve she describes, is always equal to the meridional parts of a latitude, whose complement is

equal to the latitude of the ship; and of consequence the question will be truly resolved by the following operation: The mer. parts of 10°, or of the complement of 80°=603·07, which, added to 10800, the minutes in 180°, gives 11403·07=mer. parts of 85° 50′ 48″, the complement of the latitude; consequently the ship was in latitude 4° 09′ 12″ N. when her difference of longitude was 180°. Also 603·07=the mer. parts of 10°,. or of the complement of 80°, subtracted from 3474·47=the mer. parts of 50°, or of the complement of 40°, leaves 2871·4 minutes= 47° 51′ 24″, the difference of longitude; therefore, when the ship arrived in latitude 40° N. her longitude was 60° W.—47° 51′ 24″ E.=12° 08′ 36″ W.

Corollary 1. The ship may approach infinitely near to, but can never arrive at the equator; for the meridional parts of 90° being infinite, therefore the distance of the pole from the equator, measured on the spiral line P*lofg* &c. is infinite; and consequently the corresponding difference of longitude AEDFG &c. is also infinite.

Corollary 2. The ship would revolve eternally round the pole, always approximating nearer and nearer to the equator ADCBA, which is evidently an asymptote to the spiral P*lofg* &c.

N. B. Thomas Maughan, of Quebec, the ingenious proposer of this question, remarks, that the curve described in the above solution, with a variety of its curious properties, were, about four years since, first discovered by George Baron, of New-York. He says that Mr. Baron has named this curve *the spiral of longitude*, and the meridian APC, *the polar meridian* of the spiral.

ACKNOWLEDGEMENTS, &c.

A list of the names and places of residence of the ingenious gentlemen who have given true solutions to the questions proposed in No. V. The figures annexed to the names refer to the questions answered by each, as numbered in the work.

R. Adrain, York-Town, Pennsylvania,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.
G. Baron, New-York,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.
T. Bulmer, Sunderland, England,	3.
John Capp, Harrisburg, Pennsylvania,	1, 2, 3, 4, 5, 7, 8, 11.
Ben. Cheetham, Billiard Hall,	1, 2, 5, 6, 7, 8, 9, 10, 11.
John Coope, Chester County, Pennsylvania,	2, 3, 4, 6, 7.
John Craggs, near Richmond, Virginia,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.
John D. Craig, Philadelphia,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.
Rev. T. P. Irving, Newbern, North-Carolina,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.
W. Lenhart, Baltimore,	7.
James M'Cormic, Carlisle, Pennsylvania,	4, 6, 7.
James M'Ginness, Middletown, Pennsylvania,	1, 2, 3, 4, 5, 6, 7, 8, 9, 12.
Thomas Maughan, Quebec,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.
James North, Philadelphia,	8.
R. Patterson, jun. Philadelphia,	2, 9
A. Rabbit, Harlaem, New-York,	1, 2, 3, 4, 5.
John Smithis, Philadelphia,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.
E. R. White, Danbury, Connecticut,	1, 2, 3, 4, 7, 8, 11.
Darius Yankee, Bunker's Hill,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.
T. Youle, Portland, Maine,	1, 2, 3, 4.
N. Young, Lexington, Kentucky,	1, 2, 3, 4, 5, 6, 7, 9, 11.

The Prize-Medal (value six dollars) has been awarded to *John Smithis*, Philadelphia, who is requested to authorize some person to receive the same from Mr. *G. Baron*, No. 24 Cedar-street, New-York. This medal is ready for delivery.

The non-payment of many of our country subscribers is the only obstacle which prevents us from giving gold prize-medals, value twenty dollars each.

Mr. *Capp*'s solutions arrived too late for insertion. We have to regret that want of room has compelled us to omit the elegant solutions to the prize question given by Messrs. *Adrain*, *Craig* and *Maughan*.

ARTICLE XVIII.

An Essay on the first Elements of Fluxions, being the first Section of "the Principles of Fluxions, written by the Rev. S. Vince, A. M. F. R. S. Plumian Professor of Astronomy and Experimental Philosophy in the University of Cambridge, in England." Edit. 1800. *Republished at the request of Walter Folger, jun. Nantucket.*

Definitions.

¶ 1. Every quantity is here considered as generated by motion; a line by the motion of a point; a surface by the motion of a line; a solid by the motion of a surface.*

¶ 2. The quantity thus generated is called the *fluent* or *flowing* quantity.

¶ 3. The *velocities* with which flowing quantities increase or decrease at any point of time, are called the *fluxions* of those quantities at that instant.

Cor. 1. As the velocities are in proportion to the increments or decrements *uniformly* generated in a given time, such increments or decrements will represent the fluxions.†

* Sir Isaac Newton, in the Introduction to his *Quadrature of Curves*, observes, that "these geneses really take place in the nature of things, and are daily seen in the motion of bodies. And after this manner, the ancients, by drawing moveable right lines along immoveable right lines, taught the genesis of rectangles."

† This is agreeable to Sir I. Newton's ideas on the subject. He says, "I sought a method of determining quantities from the velocities of the motions or increments with which they are generated; and calling these velocities of the motions or increments, *fluxions*, and the generated quantities *fluents*, I fell by degrees upon the method of fluxions."

Introd. to *Quad. Curves.*

Cor. 2. Hence, as any given time may be assumed, the fluxion is not an *absolute* but a *relative* quantity. When we have several cotemporary fluxions, we may assume one fluxion what we please, and thence determine the values of the others. Thus, if x and y increase uniformly, and if x increase by p in the time that y increases by q, then the cotemporary increments of x and y will be p and q, $2p$ and $2q$, $3p$ and $3q$, &c. hence, if p be assumed the fluxion of x, the fluxion of y will be q; if the former fluxion be $2p$, the latter will be $2q$, &c. &c.

Cor. 3. A *constant* quantity has no fluxion.

¶ 4. The first letters, a, b, c, &c. of the alphabet are usually put for constant quantities, and the last, v, w, x, y, z for variable ones; and they are to be thus understood, unless the contrary be expressed.

¶ 5. The fluxion of a simple quantity, as x, is expressed by placing a point over it, thus $\dot{x}$.

To find the Fluxions of Quantities.

Prop. I.

If two quantities increase or decrease uniformly, the increments or decrements generated in a given time will be as their fluxions.

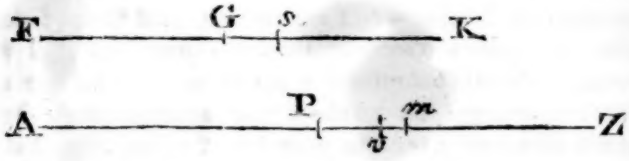
¶ 6. This appears from ¶ 3. Cor. 1.

Prop. II.

If one quantity increase uniformly, and another of the same kind increase with an accelerated or retarded velocity, and two increments be assumed which are generated in the same time; if those increments be diminished till they vanish, that ratio to which they approach as their limit is the ratio of the fluxions of those quantities.

¶ 7. Let the line FK be described with a uniform velocity, and AZ with an accelerated velocity, and

let the increments Gs, Pm be generated in the same time; let also Pv be the increments that would have



been generated in the same time, if the velocity at P had been continued uniform; then by Prop. 1. the fluxions of FK, AZ at the points G and P will be represented by Gs and Pv. Let v be the velocity with which Gs is described, and V the velocity with which Pv is described, and let the velocity at m be $V+r$; then vm is the increment which is described in consequence of the increase r of velocity since the describing point left P. Now let Pm be described with the *uniform* velocity $V+w$ in the same time that Pv and Pm are described; then it is manifest, that this uniform velocity must be between the velocities at P and at m, that is, $V+w$ is greater than V and less than $V+r$, or w is greater than 0 and less than r. Also, since the spaces described in the same time are as the velocities, $V : V+w :: Pv : Pm$. Now diminish the times in which these increments are described; then as the points v and m approach to P, Pv will continue to be described with the uniform velocity V; but r will be diminished, and by diminishing the time till it becomes indefinitely small, r will become indefinitely small; but vm is described in consequence of this increase r of velocity; hence, when r becomes indefinitely small in respect to V, the space vm must become indefinitely small in respect to Pv; therefore the ratio of $Pv : Pm$ is, in that state, indefinitely near to a ratio of equality; but it is manifest that it never can become accurately a ratio of equality, because vm will not vanish until

Pv and Pm vanish; consequently the ratio of the actual increments $Gs : Pm$ can never accurately express the ratio of the fluxions, that ratio being expressed, by the ratio $Gs : Pv$. Let us then consider what ratio $Pv : Pm$ approaches to its *limit*, when we make the time in which the increments are described, and consequently the increments themselves vanish. In every state of these increments, $V : V+w :: Pv : Pm$; and by continually diminishing the time, and consequently the increments, we diminish r, and consequently w, but V remains constant; it is manifest therefore that the ratio of $V : V+w$, and consequently that of $Pv : Pm$, continually approaches to a ratio of equality, agreeable to what we have already shown; and when the time, and consequently the increments, become actually $=0$, then $r=0$; consequently $w=0$; therefore the *limit* of the ratio of $Pv : Pm$ becomes that of $V : V$, a ratio of equality.* Hence, the *limit* of the ratio of $Gs : Pm$ is the same as the *limit* of the ratio of $Gs : Pv$, or as $Gs : Pv$, that ratio being constant; that is, *the limiting ratio of the increments is the ratio of the fluxions.*

The same is manifestly true for the limiting ratio of the decrements of two quantities; for, conceiving the describing points to move backwards, and to be retarded by the same law, the decrements sG, mP in this case become the same as the increments in the other; consequently their *limiting* ratio will express the ratio of the fluxions at G and P, or the rate at which FG, AP are, at that instant, decreasing.

Cor. 1. Hence, the *limiting* ratio of the increments or decrements of two quantities which are *both* gene-

* By keeping the ratio of the vanishing quantities thus expressed by finite quantities, it removes the obscurity which may arise when we consider the quantities themselves: this is agreeable to the reasoning of Sir I. NEWTON in his *Principia*, *lib.* 1. *sect.* 1. *lem.* 7, 8, 9.

rated by variable velocities, will be the ratio of their fluxions. And as the velocities with which these two lines increase or decrease may be made to agree with the rate of increase or decrease of any two like quantities, the proposition must be true for quantities of any kind.

Cor. 2. As the *limiting* ratio of the increments is the ratio of the fluxions, it is manifest that when the increments are in an increasing or decreasing state, the fluxions will be increasing or decreasing.

¶ 8. It has been said, that when the increments are actually vanished, it is absurd to talk of any ratio between them. It is true; but we speak not here of any ratio then existing between the quantities, but of that ratio to which they have approached as their *limit;* and that ratio still remains. Thus, let the increments of two quantities be denoted by ax^2+mx and bx^2+nx; then the *limit* of their ratio, when $x=0$, is $m : n$; for $ax^2+mx : bx^2+nx :: ax+m : bx+n ::$ (when $x=0$) $m : n$. As the quantities therefore approach to nothing, the ratio approaches to that of $m : n$ as its *limit*. Hence, if $m=n$, the *limit* of this ratio is a ratio of equality. We must therefore be careful to distinguish between the ratio of two evanescent quantities, and the *limit* of their ratio; the former ratio never arriving at the latter, as the quantities vanish at the instant that such a circumstance is about to take place.

Prop. III.

If the fluxion of x be denoted by $\dot{x}$, the fluxion of ax will be $a\dot{x}$.

¶ 9. For if x increase uniformly, ax will also increase uniformly, and a times as fast: hence, by Prop. 1, the fluxion of the latter will be a times greater than that of the former, or it will be $a\dot{x}$.

Cor. Hence, in taking the fluxion of a variable quantity multiplied into a constant one, the constant multiplier is retained.

Prop. IV.

The fluxion of $x \pm a$ *is* $\dot{x}$.

¶ 10. For a being constant, and only connected to x by the signs + or −, it does not affect the *increase* or *decrease* of the quantity; therefore the fluxion is the same as the fluxion of x, or it is $\dot{x}$.

Cor. Hence, constant quantities connected to variable ones by the signs + or −, disappear when the fluxions are taken.

Prop. V.

Given $(\dot{x})$ *the fluxion of* x, *to find the fluxion of* x^n, *n being a whole number.*

¶ 11. Let x increase uniformly by v and become $x+v$, then will x^n become $\overline{x+v}^n$; but by the well known Elements of Algebra, $\overline{x+v}^n = x^n + nx^{n-1}v + n.\frac{n-1}{2}x^{n-2}v^2 + \&c.$ and if from this quantity we take x^n, there remains $nx^{n-1}v + n.\frac{n-1}{2}x^{n-2}v^2 + \&c.$ for the cotemporary increment of x^n; but although x increase uniformly by v, x^n does not increase uniformly; for if in the increment of x^n we substitute 1, 2, 3, &c. for v, and take the differences of the results, these differences will not be equal; hence, to get the ratio of the fluxion of x to the fluxion of x^n we must, according to Prop. 2. take the *limiting* ratio of the increments. Now the increment of x : the increment of x^n :: v : $nx^{n-1}v + n.\frac{n-1}{2}x^{n-2}v^2 + \&c.$ :: 1 : $nx^{n-1} + n.\frac{n-1}{2}x^{n-2}v + \&c.$ and to get the *limiting* ratio of these increments, we

must make $v=0$, in which case the ratio becomes $1 : nx^{n-1}$, which therefore expresses the ratio of the fluxion of x to the fluxion of x^n; but $\dot{x}$ denotes the fluxion of x, therefore $nx^{n-1}\dot{x}$ represents the cotemporary fluxion of x^n.

If $n=0$, $x^n=1$ a constant quantity; therefore by ¶ 3. Cor. 3. it has no fluxion.

Prop. VI.

To find the fluxion of $x^{\frac{n}{m}}$, *m and n being any whole numbers.*

¶ 12. Put $y=x^{\frac{n}{m}}$, then $y^m=x^n$; hence, by taking the fluxions, $my^{m-1}\dot{y}=nx^{n-1}\dot{x}$, $\therefore \dot{y}=\frac{nx^{n-1}\dot{x}}{my^{m-1}}=$ (by substituting for y its value in terms of x) $\frac{nx^{n-1}\dot{x}}{mx^{\frac{mn-n}{m}}}=\frac{nx^{n-1}\dot{x}}{mx^{n-\frac{n}{m}}}=\frac{n}{m}\times x^{\frac{n}{m}-1}\dot{x}$.

Cor. Let the root be a compound quantity as a^m+x^m, to find the fluxion of $\overline{a^m+x^m}^{\frac{1}{n}}$. Put $y=\overline{a^m+x^m}^{\frac{1}{n}}$, then $y^n=a^m+x^m$, and $ny^{n-1}\dot{y}=mx^{m-1}\dot{x}$, hence, $\dot{y}=\frac{mx^{m-1}\dot{x}}{ny^{n-1}}=\frac{mx^{m-1}\dot{x}}{n\times\overline{a^m+x^m}^{\frac{n-1}{n}}}=\frac{1}{n}\times\overline{a^m+x^m}^{\frac{1-n}{n}}\times mx^{m-1}\dot{x}=\frac{1}{n}\times\overline{a^m+x^m}^{\frac{1}{n}-1}\times mx^{m-1}\dot{x}$.

(To be continued.)

ARTICLE XIX.

New Questions to be solved in the next Number.

I. Quest. 52. *By Henry Smith, New-York.*

Required a general rule for reducing Sterling money into Federal currency; and also a general rule for reducing Federal currency into Sterling money.

II. Quest. 53. *By A. Rabbit, Harlaem, New-York.*

Required a more elegant theorem than that given by Mr. Shepherd, in page 203 of his Columbian Accountant, for finding the side of what he calls the greatest inscribed square of an equilateral triangle.

III. Quest. 54. *By Ben. Cheetham, Billiard-Hall.*

In a single throw with two dice, on what particular number is it most advantageous to bet? and what is the chance of throwing that number?

IV. Quest. 55. *By Diarius Yankee.*

By selling a piece of muslin at a certain price per yard I gained the prime cost of nine yards, which was just as much per cent. as the number of yards in the piece; required the number of yards.

V. Quest. 56. *By James M'Cormic, Carlisle, Pennsylvania.*

The area of a field in the form of a plane rectilineal triangle is a, and the lengths of the two least sides are b and c; what is the length of the third or greatest side?

VI. Quest. 57. *By the Rev. Thomas P. Irving, Newbern, North-Carolina.*

It is required to describe a circle that shall pass through an angular point of a given square, and touch two sides of the same square.

VII. QUEST. 58. *By John D. Craig, Philadelphia.*

In any right angled plane triangle, if a straight line be drawn from the right angle, perpendicular to the hypotenuse; and from the point where this perpendicular intersects the hypotenuse, a second straight line be drawn perpendicular to the base; and from the point where this perpendicular meets the base, a third straight line be drawn perpendicular to the hypotenuse, &c. &c. *in infinitum;* I say, the difference between the hypotenuse and the base, is to the base as the perpendicular of the triangle is to the sum of all the other perpendiculars; required a demonstration.

VIII. QUEST. 59. *By W. Lenhart, Baltimore.*

In a plane rectilineal triangle are given the rectangle of the two sides, the rectangle of the segments of the base made by a perpendicular let fall from the vertical angle, and the area; to find the sides of the triangle by a more elegant method than that used by Mr. Emerson in Prob. 87. book 2. of his Algebra.

IX. QUEST. 60. *By John D. Craig.*

One night in the present year the true amplitude of a star of the first magnitude, at rising, was E. 26° 44′ N. Some hours after its true altitude was 24° 26′, and four hours fifty-six minutes and twenty-three seconds after that I observed its transit over the meridian: quere the star and the latitude of the place of observation?

X. PRIZE QUEST. 61. *By Robert Adrain.*

To determine the nature of the catenaria volvens, or the figure which a perfectly flexible chain of uniform density and thickness will assume, when it revolves with a constant angular velocity about an axis, to which it is fastened at its extremities, in free and nongravitating spaces.

ARTICLE XX.

Solutions of the Questions proposed in Article XIX.

I. Quest. 52. *Solved by E. R. White, Danbury, Connecticut.*

Let s denote any sum sterling, expressed in pounds, or by means of the decimal of a pound; and f an equivalent sum of federal currency. Then because 9*l.* sterling are equal in value to 40 *dolls.* we have $9 : 40 :: s : f$; and by art. 1. §. 17. Math. Corr. $36 : 40$ or $9 : 10 :: 4s : f$: hence by page 30 and 31 Math. Corr. the following theorems are evident.

Theorem 1. If any sum of sterling money, expressed in the denomination of pounds, be multiplied by 4, the product increased by $\frac{1}{9}$ of itself will be the equivalent federal money.

Theorem 2. Any sum of federal currency, diminished by $\frac{1}{10}$ of itself, leaves four times the sterling money, expressed in the denomination of pounds.

Examples.

£ 236 16*s.* 9¾*d.*=£ 236·840625 sterling.

4

9) 947·3625

105·2625

1052 *dolls.* 62·5 *cts.*=1052·625 *dolls.*

10) 1052·625 *dolls.*

105·2625

4) 947·3625

£ 236·840625=236 *l.* 16 *s.* 9¾ *d.* sterling.

II. QUEST. 53. *Solved by the proposer, A. Rabbit.*

The contents of page 203 of the Columbian Accountant, sung to the tune of "*the pump* of Lake Champlaine,*" will completely exhibit the stupidity of Shepherd's rule mentioned in the question. With this remark I proceed to develope the theorem sought. Let s = the side of the equilateral triangle, p = a straight line drawn from the opposite angle perpendicular to the side s, and x = the side of the inscribed square. Then it is well known that in every plane rectilineal triangle, $x = \frac{ps}{p+s}$; but in the equilateral triangle $p = \frac{1}{2}s\sqrt{3}$; and this written for p in the former equation gives $x = \frac{\frac{1}{2}s^2\sqrt{3}}{\frac{1}{2}s\sqrt{3}+s} = \frac{\sqrt{3}}{2+\sqrt{3}} \times s = (2\sqrt{3} - 3) \times s = s \times .4641016151$ &c. the theorem required. "The *greatest* inscribed square of a triangle" is absurd; but it clearly proves Shepherd to be ignorant of Geometry: this is evident from def. 1. book 4. Euclid, and the nature of a triangle.

III. QUEST. 54. *Solved by B. Cheetham, Billiard-Hall.*

Let the two first horizontal lines in the annexed scheme represent the figures on the 12 faces of the two dice; then by combining each figure in the first line, with every figure in the second, it will be evident that two dice may be thrown 36 different ways; and that there

1	2	3	4	5	6
1	2	3	4	5	6
2	3	4	5	6	7
3	4	5	6	7	8
4	5	6	7	8	9
5	6	7	8	9	10
6	7	8	9	10	11
7	8	9	10	11	12

* The theological properties of this wonderful pump have, I understand, been sufficiently investigated by the members of a certain church in New-York; and I promise to unfold the mathematical principles of the same, in some future Number. A. R.

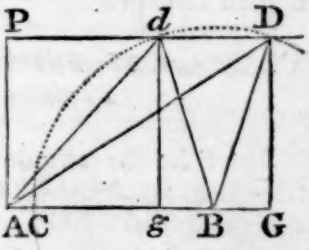
is one chance for throwing either 2 or 12, two chances for throwing either 3 or 11, three chances for throwing either 4 or 10, four chances for throwing either 5 or 9, five chances for throwing either 6 or 8, and 6 chances for throwing 7. Hence 7 is the number required; and the chances for throwing 7 are 6, and those against throwing it are 30.

IV. QUEST. 55. *Solved by Thomas I. S. Singleton, Newbern, North-Carolina.*

Put x=the number of yards, and y=the prime cost of a yard; then $9y$=the prime cost of 9 yards, and xy=the prime cost of the piece. But by the question $100 : x :: xy : 9y :: x : 9$; whence $x^2=900$, and $x=30$, the number of yards required.

V. QUEST. 56. *Solved by Thomas Maughan, Quebec.*

Let the straight line AB=b not less than BC=c, be considered as the base of the triangle. Draw AP at right angles to AB, and equal to $\frac{2a}{b}$, which by mensuration is =the altitude of the triangle,



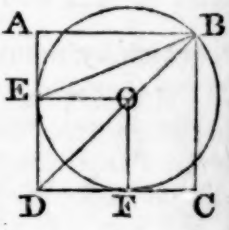
and consequently cannot be greater than BC. Through P draw PD parallel to AB; on the centre B with the radius BC, describe the arc CdD, intersecting PD in D, and join AD and DB; then ADB is the triangle in question, and AD is the greatest or third side sought. This is too evident to require a demonstration.

From D draw DG at right angles to AB; then in the oblique angled triangle ABD, are given AB=b,

BD=BC=c and DG=AP=$\frac{2a}{b}$; hence by prop. 47. book 1. Euc. BG=$\sqrt{c^2-\frac{4a^2}{b^2}}=\frac{\sqrt{b^2c^2-4a^2}}{b}$. Further, when the perpendicular falls without the triangle ABD, by prop. 12. book 2. Euclid, $\overline{AD}^2=b^2+c^2+2\sqrt{b^2c^2-4a^2}$; but when the perpendicular falls within this triangle, by prop. 13. book 2. Euclid, $\overline{AD}^2=b^2+c^2-2\sqrt{b^2c^2-4a^2}$; which last value of $\overline{AD}^2$ is evidently greater than that of $\overline{AB}^2$, when $2\sqrt{b^2c^2-4a^2}$ is less than c^2; and in this case the question admits of two answers.

Corollary. When $2\sqrt{b^2c^2-4a^2}$ is greater than c^2, the greatest side AD is limited to one value; and when b^2c^2 is less than $4a^2$, that is, when bc is less than $2a$, the problem is impossible. Also when $bc=2a$, the third side AD is the hypotenuse of a right angled triangle.

VI. QUEST. 57. *Solved by the Rev. Thomas P. Irving, Newbern, North-Carolina.*



Let B be the angular point of the given square ABCD. Draw the diagonal BD; with the straight line BE, intersecting AD in E, bisect the angle ABD; and thro' E draw the straight line EO parallel to AB, cutting BD in O: then is O the centre, and OE the radius of the required circle.

Demonstration.

Draw the straight line OF parallel to AD. Because EO is parallel to AB, the angle BEO=the

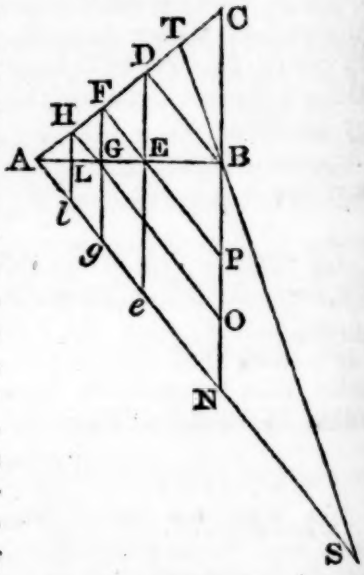
alternate angle ABE,=EBO by construction, and therefore BO = OE. Moreover, by construction, OEDF being a parallelogram about the diagonal of a square, is well known to be a square; consequently OF=OE,=OB by what has just been proved. Also OFD, and OED, being angles of a square, are right angles. Hence, a circle described about the centre O with the radius OE, will pass through the angular point B, and touch the two sides AD and DC in the points E and F: as was to be done.

Corollary 1. Put AB=AD=s the side of the given square, and BO=OE=OF=r the radius of the required circle; then the diagonal BD= $s\sqrt{2}$. Further, since BE bisects the angle ABD, AB: BD:: AE: ED, by prop. 3. book 6. Euc. and by composition AB+BD: BD:: AE+ED=AD: ED: that is $s+s\sqrt{2} : s\sqrt{2} : : s : r=\frac{\sqrt{2}}{1+\sqrt{2}}\times s=(2-\sqrt{2})\times s=$ $s\times$·5857864376.

Otherwise. OD=$r\sqrt{2}$, and BD=$r\sqrt{2}+r=s\sqrt{2}$: hence $r=\frac{\sqrt{2}}{\sqrt{2}+1}\times s=(2-\sqrt{2})\times s$, the same as above.

Corollary 2. The difference between twice the side of the given square and its diagonal, is equal to the radius of the required circle. For BD=$s\sqrt{2}=d$; and by cor. 1. $r=(2-\sqrt{2})s=2s-s\sqrt{2}==2s-d$. From this property another method of describing the required circle is easily inferred.

VII. QUEST. 58. *Solved by Robert Adrain, Reading, Pennsylvania.*



Let ABC be any right angled plane triangle, having the perpendiculars BD, DE, EF, FG, &c. drawn as described in the question. Draw AN at right angles to AC, meeting CB produced in N; and let all the perpendiculars, just mentioned, be produced to meet AN and BN in *e*, *g*, *l*, O, P, &c. &c. Because DN, F*e*, H*g*, &c. are parallelograms, the sum of all the perpendiculars on AC, viz. BD+EF+GH+&c.=N*e*+*eg*+*gl*+&c. *ad infinitum*=AN. For a similar reason the sum of DE, FG, HL, &c. *ad infinitum*=BN. The sum of all the perpendiculars, BD+DE+EF+FG+&c. is therefore equal to AN+BN. Produce now AN to S, so that NS may be equal to BN; then AN+BN=AS. Join SB and produce it to meet AC in T, and we easily discover that the triangles ABS and CTB are similar, and that AT=AB: whence TC: AB :: CB: AS, or AC—AB: AB: CB: AS, which was to be demonstrated.

The same, solved by John Capp, Harrisburg, Pennsylvania.

Let ABC be any right angled plane triangle, in which the perpendiculars BD, DE, EF, FG, &c.

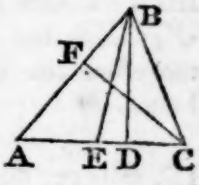
in infinitum, are drawn according to the question. The right angled triangles CDB, BED, DEF, FEG, &c. are evidently similar to each other, and to the original triangle ABC; and hence AC : AB : : CB : BD : : BD : DE : : DE : EF : : EF : FG : : &c. But by prop. 12. book 5. Euc. AC : AB : : CB+BD+DE+EF+&c. : BD+DE+EF+FG+&c.=s; and by division, AC—AB : AB : : CB : s=the sum of all the perpendiculars BD, DE, EF, FG, &c. *in infinitum:* as was to be demonstrated.

Corollary 1. CB : AB : : AC+AB : s.

Corollary 2. AC—AB : AC : : CB : s+CB.

Corollary 3. CB : AC : : AC+AB : s+CB.

VIII. QUEST. 59. *Solved by the proposer, W. Lenhart, Baltimore.*



Let ABC be the plane rectilineal triangle. Draw BD perpendicular to AC, and CF perpendicular to AB; also from B to E, the middle point of AC, draw BE. Put the given rectangle AB.BC=b, the given rectangle AD.DC=c, twice the given area of ABC=a; and BC=x; then is AB=$\frac{b}{x}$. Let s=the natural sine of the angle ABC, to the radius =1; then by mensuration AB.BC.s=a: consequently $s=\frac{a}{b}$, CF=sx, BF=$x\sqrt{1-s^2}$, and AB.BF=$b\sqrt{1-s^2}$=by question 46. Math. Corr. $\overline{BE}^2 \sim \overline{EC}^2 = \overline{BE}^2 \sim (\overline{ED}^2 + AD.DC) = \overline{BD}^2 - c$, or $c-\overline{BD}^2$. Hence $\overline{BD}^2 = c+b\sqrt{1-s^2}$, or $\overline{BD}^2 = c-b\sqrt{1-s^2}$; and by restoring the value of s we have $\overline{BD}^2 = c+\sqrt{b^2-a^2}$, when the angle ABC is acute, and $\overline{BD}^2$

$=c-\sqrt{b^2-a^2}$ when ABC is obtuse. The value of BD being thus determined, the three sides of the triangle are easily found.

The same, solved by Robert Adrain, Reading, Pennsylvania.

Let ABC be the plane rectilineal triangle. Draw BD and CF perpendicular to AC and AB respectively; also from B draw BE to E the middle point of AC. Put twice the given area$=a=$AB.FC, the given rectangle AB.BC$=b$, and the other given rectangle AD.DC$=c$. Now $\overline{AB}^2.\overline{BC}^2=b^2$, and $\overline{AB}^2.\overline{FC}^2=a^2$; wherefore $\overline{AB}^2.\overline{BC}^2-\overline{AB}^2.\overline{FC}^2=\overline{AB}^2.\overline{BF}^2=b^2-a^2$, and AB . BF$=\sqrt{b^2-a^2}=$, by question 46. Math. Corr. $\overline{BE}^2$ or $\overline{EC}^2=(\overline{BD}^2+\overline{DE}^2)$ or $\overline{EC}^2=\overline{BD}^2-$AD.DC or AD.DC$-\overline{BD}^2=\overline{BD}^2-c$ or $c-\overline{BD}^2$. Hence if the angle ABC be acute $\overline{BD}^2=c+\sqrt{b^2-a^2}$, but if obtuse $\overline{BD}^2=c-\sqrt{b^2-a^2}$: and therefore the sides of the triangle may be easily found.

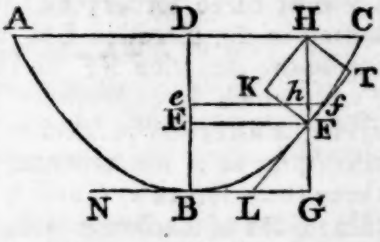
IX. QUEST. 60. *Solved by the proposer, John D. Craig, Philadelphia.*

In transcribing this question, I inadvertently wrote 24° 26′ for the true altitude, *which ought to have been* 24° 36′.

The star's distance from the meridian, in solar time, is 4*h*. 56 *min*. 23 *sec*.$=$4*h*. 57 *min*. 11 *sec*. 42·6 *thds*. siderial time, which corresponds with 74° 17′ 56·6″ the star's horary angle. Let radius$=1$, $c=$ ·2706163 the natural cosine of the star's horary angle, $a=$·4498387 the natural sine of the amplitude 26°

44′, b=·4162808 the natural sine of the altitude 24° 36′, and x=the natural cosine of the latitude. Then by spherics $1 : a :: x : ax$=the natural sine of the star's declination; and $\sqrt{1-a^2x^2}$=the natural sine of the star's polar distance. Lastly by the theorem in case 8. of oblique spherical triangles, sec. 4. book 3. Emerson's Trigonometry, $cx\sqrt{1-a^2x^2}+ax\sqrt{1-x^2}=b$ or $cx\sqrt{1+ax}\times\sqrt{1-ax}+ax\sqrt{1+x}\times\sqrt{1-x}=b$; from which, by means of a table of logarithms, we readily approximate to x=·7678312=the natural cosine of the latitude 39° 50′ 26″: consequently ax =·3454002=the natural sine of 20° 12′ 22″ N. answering nearly to the declination of the star Arcturus.

X. Prize Quest. 61. *Solved by the proposer, Robert Adrain, Reading, Pennsylvania.*



Let ABC be the chain revolving uniformly about the fixed axis AC, to which it is fastened at its extremities A and C. Because the chain is supposed to retain the same constant figure during its uniform revolution, it must evidently lie in a plane passing along the axis AC, and dispose itself into the form of a curve in that plane, the nature of which I shall endeavour to unfold.

Let B be the vertex of the curve, or the point where the tangent NBG is parallel to AC, and from B let fall on AC the perpendicular BD. Suppose BF to be any variable part of the curve beginning at B; through F draw FE, GH, at right angles to BD, DC, also TFL the tangent at F meeting NG in L;

on the diagonal FH complete the rectangle KT; and let ef be infinitely near and parallel to EF, meeting FH in h, and making Ee or Fh, hf, and Ff the fluxions of BE, EF, and BF, respectively.

Put BD$=a$, BE$=x$, EF$=y$, BF$=z$, Ee=F$h=\dot{x}$, $hf=\dot{y}$, F$f=\dot{z}$, also put n=the constant tension of the chain at B, t=the tension at F, f=the centrifugal force of BF, which is produced by its circular motion about DH, and which is equal to the sum of the centrifugal forces of all its particles in directions parallel to BD, c=the celerity of the point B in the circumference of a circle about the centre D, m=the quantity of matter in the length 1 of the chain; moreover, put $\frac{mc^2}{a^2}=r$, $\frac{4n}{r}=b^2$, and assume $g=32\frac{1}{6}$ feet to measure the common force of gravity.

The part of the chain BF is supported in equilibrio by three forces; by the constant tension at B acting in the direction BN, by the tension at F acting in the direction FT, and by the centrifugal force f with which it endeavours to recede from the axis DH in a direction parallel to BD or GH: therefore, according to a fundamental principle of mechanics these three forces n, t, and f, are proportional to the three sides of a triangle, GL, LF, FG, drawn parallel to their lines of direction, and LGF is a right angle, consequently $n^2+f^2=t^2$. Further, because the triangles FGL, Fhf, are similar, it appears that n, f, and t are also proportional to $\dot{y}$, $\dot{x}$ and $\dot{z}$, or to the radius, tangent, and secant of the angle efL whence $\dot{y}=\frac{n\dot{x}}{f}$, and $\dot{z}=\frac{t\dot{x}}{f}$. And we may remark, that these conclusions are equally true in all catenarian curves of which the particles are urged by any forces in parallel directions.

Again, the accelerative force of gravity being expressed by g, the centrifugal force of a particle at B is truly measured by $\frac{c^2}{a}$. See Simp. Flux. Art. 211. And, because all the particles in BC revolve about AC in the same periodic time, it is well known that their centrifugal forces are directly as the radii of the circles they describe; therefore as BD : FH, or as $a : a-x :: \frac{c^2}{a} : \frac{c^2}{a^2} \times \overline{a-x}$ = the centrifugal force of a particle at F: and by hypothesis the quantity of matter in Ff is $m\dot{z}$, consequently $\frac{c^2 m}{a^2} \cdot \dot{z} \times \overline{a-x} = r\dot{z} \times \overline{a-x} = \dot{f}$. The force $\dot{f}$ acting in the direction HF, and expressed by it, is, by the resolution of forces, equivalent to the two forces KF, TF. The force KF being at right angles to Ff cannot possibly make the tension greater at f than at F, but only produces the curvature of Ff, the force TF is solely employed in augmenting the tension at f, and is therefore $= \dot{t}$; whence as FH : FT, that is as $\dot{z} : \dot{x} :: \dot{f} = r\dot{z} \times \overline{a-x} : \dot{t} = r\dot{x} \times \overline{a-x} = ra\dot{x} - r\dot{x}x$, and the fluent $t = rax - \frac{1}{2}rx^2$; but when $x=0$, $t=n$, therefore the correct fluent is $t = n + rax - \frac{1}{2}rx^2$, and consequently $\dot{y} = \frac{\dot{x}}{f} = \frac{\dot{nx}}{\sqrt{t^2 - n^2}}$

$$= \frac{\dot{nx}}{\sqrt{n + rax - \frac{1}{2}rx^2 - n^2}} = \frac{\frac{2n}{r}\dot{x}}{\sqrt{\frac{4n}{r} + 2ax - x^2 . 2ax - x^2}}, \text{ that}$$

is $\dot{y} = \frac{\frac{1}{2}b^2\dot{x}}{\sqrt{b^2 + 2ax - x^2} \times 2ax - x^2}$, and $\dot{z} = \frac{\dot{x}}{f} =$

$\frac{\frac{1}{2}b^2\dot{x}+2a\dot{x}x-\dot{x}x^2}{\sqrt{b^2+2ax-x^2}\times 2ax-x^2}$. But let us pursue this subject farther, and determine the relations of x, y, and z. The values of y and z are not indeed assignable by x in finite algebraic terms, nor even by the areas of conic sections; but we shall obtain them by having recourse to their arcs. For this purpose assume $v^2=2ax-x^2$, and the values of $\dot{y}$ and $\dot{z}$ are transformed into $\dot{y}=\frac{\frac{1}{2}b^2\dot{v}}{\sqrt{a^2-v^2}\times b^2+v^2}$, and $\dot{z}=\frac{\frac{1}{2}b^2\dot{v}+v^2\dot{v}}{\sqrt{a^2-v^2}\times b^2+v^2}$. By addition, $\dot{z}+\dot{y}=\frac{\dot{v}\times b^2+v^2}{\sqrt{a^2-v^2}\times b^2+v^2}=\dot{v}\sqrt{\frac{b^2+v^2}{a^2-v^2}}$. Now $\frac{a\dot{v}}{b}\sqrt{\frac{b^2+v^2}{a^2-v^2}}=\dot{E}$ is known to be the fluxion of an elliptic arc E, reckoning from the vertex of the transverse, the corresponding ordinate on the transverse being v, and the semitransverse and semiconjugate axes $\frac{a\sqrt{a^2+b^2}}{b}$, and a, respectively. Therefore $\dot{z}+\dot{y}=\frac{b\dot{E}}{a}$, and the fluent is $z+y=\frac{bE}{a}$, which needs no correction.

Again, by subtraction, $\dot{z}-\dot{y}=\frac{v^2\dot{v}}{\sqrt{a^2-v^2}\times b^2+v^2}$. To exhibit the fluent of which, let there be an hyperbola having the semitransverse and semiconjugate axes a and b, the arc between the vertex and any point=H, the perpendicular from the centre on the tangent at that point=v, and the intercepted tangent =T: and by investigating the value of $\dot{T}-\dot{H}$ in

terms of v, we shall find it $= \frac{-v^2\dot{v}}{\sqrt{a^2-v^2}\times\sqrt{b^2+v^2}}$. Consequently $\dot{z}-\dot{y}=-\dot{T}+\dot{H}$, and taking the fluent we have $z-y=-T+H$. But this fluent must be corrected, for when $z-y=0$, then $v=0$, therefore the tangent T coincides with the assymptote, and $T-H=D$, D being the excess of the assymptote above the curve, when both are produced indefinitely. For a computation of which see Simp. Flux. art. 435. The correct equation of the fluents therefore is $z-y=D-T+H$; whence $z=\frac{bE}{2a}+\frac{D-T+H}{2}$, and $y=\frac{bE}{2a}-\frac{D-T+H}{2}$, and when $v=a$, then $T-H=0$, in which case we obtain the whole curve $ABC=\frac{bE}{a}+D$, and the axis $AC=\frac{bE}{a}-D$.

If the axis AC and the curve ABC be very nearly in the ratio of equality, or, which is the same in effect, if a be supposed incomparably less than b; the value of $\frac{b}{\sqrt{b^2+v^2}}$ becomes indefinitely near to unity: in which circumstance $\dot{y}=\frac{1}{2}b\times\frac{\dot{v}}{\sqrt{a^2-v^2}}$, and consequently $y=\frac{1}{2}b\times$ circular arc A to the sine $\frac{v}{a}$, and radius unity. Whence it appears, that in this limiting case, our new curve coincides with the celebrated *harmoniac curve*, the nature of which was first discovered by *Dr. Brook. Taylor.*

ACKNOWLEDGEMENTS, &c.

A list of the names and places of residence of the ingenious gentlemen who have given true solutions to questions proposed in No. VI. The figures annexed to the names refer to the questions answered by each, as numbered in the work.

R. Adrain, Reading, Pennsylvania,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10.
G. Baron, New-York,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10.
John Capp, Harrisburg, Pennsylvania,	1, 2, 4, 5, 6, 7, 8, 9.
Ben. Cheetham, Billiard Hall,	1, 2, 3, 4, 5.
Geo. W. Cook, Newbern, North-Carolina,	1, 3, 4, 6.
John Coope, Chester County, Pennsylvania,	2, 4, 5, 6, 7, 8.
John Craggs, near Richmond, Virginia,	1, 2, 3, 4, 5, 6, 7, 8, 9.
John D. Craig, Philadelphia,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10.
Martin Gibson, George-Town, South-Carolina,	1, 2, 4, 5, 6, 7, 8.
Rev. T. P. Irving, Newbern, North-Carolina,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10.
William Lenhart, Baltimore,	1, 2, 4, 5, 6, 7, 8.
James M'Cormic, Carlisle, Pennsylvania,	1, 2, 4, 6.
James M'Ginness, Harrisburg, Pennsylvania,	1, 2, 4, 5, 6, 7, 8, 9.
Thomas Maughan, Quebec,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10.
A. Rabbit, Harlaem New-York,	1, 2, 3, 4, 5, 6, 7, 8, 9.
Seth Robson, Savannah, Georgia,	1, 2, 3, 4, 5, 6, 7, 8, 9.
Thomas I. S. Singleton, Newbern, North-Carolina,	1, 2, 3, 4, 6.
Henry Smith, New-York,	1, 2, 3, 4, 5, 6, 7.
John Smithis, Philadelphia,	1, 2, 3, 4, 5, 6, 7, 8, 9.
Peter Spangler, York-Town, Pennsylvania,	4, 5, 8.
E. R. White, Danbury, Connecticut,	1, 2, 4, 5, 6.
Thomas Whittaker, York-Town, Pennsylvania,	1, 2, 4, 5, 6, 7.
Diarius Yankee, Bunker's Hill,	1, 2, 3, 4, 5, 6, 7, 8, 9.
T. Youle, Portland, District of Maine,	1, 2, 4, 6.
N. Young, Lexington, Kentucky,	1, 2, 4, 5, 6, 7, 8, 9.

The Prize-Medal (value six dollars) has been awarded to *Robert Adrain*, Reading, Pennsylvania; and will be delivered, by G. *Baron*, No. 24 Cedar-street, New-York, to any person authorized to receive the same.

A. Rabbit will not, in any future Number, be permitted to propose questions concerning the blunders of stupid Shepherd: we had rather soar aloft with the Eagle than waddle in mud with a Goose.

Some of our country subscribers have discovered, that the best way of encouraging the mathematics, is never to remit us a single cent for our Numbers. This new idea is honourable to those who discovered it: posterity would certainly thank us, were we to publish a list of their names.

ARTICLE XVIII.

Continued from page 138.

¶ 13. Hence it appears, that whether the root be a simple or a compound quantity, the fluxion of any power thereof is found by the following

RULE:

Multiply by the index, diminish the index by unity, and multiply by the fluxion of the root.

Examples.

Ex. 1. The fluxion of x^9 is $9x^8\dot{x}$.

Ex. 2. The fluxion of $3y^5$ is $15y^4\dot{y}$.

Ex. 3. The fluxion of $\frac{3}{2}y^{\frac{4}{7}}$ is $\frac{12}{14}y^{-\frac{3}{7}}\dot{y} = \frac{6\dot{y}}{7y^{\frac{3}{7}}}$.

Ex. 4. The fluxion of $\frac{5}{9}x^{\frac{7}{11}}$ is $\frac{35}{99}x^{-\frac{4}{11}}\dot{x} = \frac{35\dot{x}}{99x^{\frac{4}{11}}}$

Ex. 5. The fluxion of $\frac{4}{7}x^{\frac{11}{9}}$ is $\frac{44}{63}x^{\frac{2}{9}}\dot{x}$.

Ex. 6. What is the fluxion of $\overline{a^2+x^2}^3$?

Here the root is a^2+x^2, and its fluxion $2x\dot{x}$; hence, the fluxion required is $3\times\overline{a^2+x^2}^2\times 2x\dot{x} = \overline{a^2+x^2}^2\times 6x\dot{x}$.

Ex. 7. What is the fluxion of $\sqrt{a^2+x^2}$, or of $\overline{a^2+x^2}^{\frac{1}{2}}$?

Here the root is a^2+x^2, and its fluxion $2x\dot{x}$; hence, the fluxion is $\frac{1}{2}\times\overline{a^2+x^2}^{-\frac{1}{2}}\times 2x\dot{x} = \frac{x\dot{x}}{\overline{a^2+x^2}^{\frac{1}{2}}}$.

Ex. 8. What is the fluxion of $\overline{x^2+y^2}^{\frac{3}{2}}$?

Here the root is x^2+y^2, and its fluxion $2x\dot{x}+2y\dot{y}$; hence, the fluxion required is $\frac{3}{2}\times\overline{x^2+y^2}^{\frac{1}{2}}\times\overline{2x\dot{x}+2y\dot{y}}$ $=3\times\overline{x^2+y^2}^{\frac{1}{2}}\times\overline{x\dot{x}+y\dot{y}}$.

Ex. 9. What is the fluxion of $\overline{x+y}^2$?

Here the root is $x+y$, and its fluxion $\dot{x}+\dot{y}$; hence, the fluxion required is $2\times\overline{x+y}\times\overline{\dot{x}+\dot{y}}$.

Ex. 10. What is the fluxion of $\overline{a^5+x^5}^{\frac{1}{2}}$?

Here the root is a^5+x^5, and it's fluxion $5x^4\dot{x}$; hence, the fluxion required is $\frac{1}{2}\times\overline{a^5+x^5}^{-\frac{1}{2}}\times5x^4\dot{x}$ $=\frac{5x^4\dot{x}}{2\times\overline{a^5+x^5}^{\frac{1}{2}}}$

Ex. 11. What is the fluxion of $\frac{1}{\overline{a^2+x^2}^{\frac{5}{9}}}$?

This quantity becomes $\overline{a^2+x^2}^{-\frac{5}{9}}$, and the root is a^2+x^2, whose fluxion is $2x\dot{x}$; hence, the fluxion required is $-\frac{5}{9}\times\overline{a^2+x^2}^{-\frac{14}{9}}\times2x\dot{x}=\frac{-10x\dot{x}}{9\times\overline{a^2+x^2}^{\frac{14}{9}}}$. In like manner, bring any quantity from the denominator up to the numerator, by changing the sign of the index, and then proceed by the rule.

Ex. 12. What is the fluxion of $\overline{ax^2+by^3+cz^4}^{\frac{7}{3}}$?

Here the root is $ax^2+by^3+cz^4$, and its fluxion $2ax\dot{x}+3by^2\dot{y}+4cz^3\dot{z}$; hence, the fluxion required is $\frac{7}{3}\times\overline{ax^2+by^3+cz^4}^{\frac{4}{3}}\times\overline{2ax\dot{x}+3by^2\dot{y}+4cz^3\dot{z}}$.

Ex. 13. What is the fluxion of $\sqrt{x^2+\sqrt{a^2+y^2}}$?

Put $z=\sqrt{x^2+\sqrt{a^2+y^2}}$, then $z^2=x^2+\sqrt{a^2+y^2}$; now the fluxion of $\sqrt{a^2+y^2}$, or of $\overline{a^2+y^2}^{\frac{1}{2}}$, is $\frac{1}{2}\times\overline{a^2+y^2}^{-\frac{1}{2}}\times2y\dot{y}=\overline{a^2+y^2}^{-\frac{1}{2}}\times y\dot{y}$; hence, $2z\dot{z}=2x\dot{x}$

$+\overline{a^2+y^2}\Big)^{-\frac{1}{2}} \times \dot{y}\dot{y}$, therefore $\dot{z} = \dfrac{2x\dot{x}+\overline{a^2+y^2}\Big)^{-\frac{1}{2}}.\dot{y}\dot{y}}{2z}$

$= \dfrac{2x\dot{x}+\overline{a^2+y^2}\Big)^{-\frac{1}{2}}\dot{y}\dot{y}}{2\sqrt{x^2+\sqrt{a^2+y^2}}}$

Prop. VII.

To find the fluxion of a product xy.

¶ 14. The fluxion of $\overline{x+y}^2$, by the last rule, is $2\times x+y\times\dot{x}+\dot{y}=2x\dot{x}+2x\dot{y}+2y\dot{x}+2y\dot{y}$; also, $\overline{x+y}^2 = x^2+2xy+y^2$, whose fluxion is $2x\dot{x}+$ *the fluxion of* $2xy+2y\dot{y}$; make these two values of the fluxion of $\overline{x+y}^2$ equal to each other; omit the first and last terms which are common to both, and we have *the fluxion of* $2xy=2x\dot{y}+2y\dot{x}$; hence the fluxion of xy is $x\dot{y}+y\dot{x}$.

Otherwise thus. If we suppose x constant, the fluxion of xy is $x\dot{y}$ by Prop. 3; and if we suppose y constant, the fluxion is $y\dot{x}$; hence, if neither be constant, the fluxion is $x\dot{y}+y\dot{x}$.

Cor. Hence, we may find the fluxion of xyz. For if $v=xyz$, and $w=xy$, then $v=wz$, and $\dot{v}=w\dot{z}+z\dot{w}$; but $w=xy$, $\therefore \dot{w}=x\dot{y}+y\dot{x}$; substitute these values for w and $\dot{w}$, and we get $\dot{v}=xy\dot{z}+zx\dot{y}+zy\dot{x}$.

¶ 15. In like manner we proceed for any number of factors; hence, the fluxion of the product of any number of quantities is found by the following

RULE:

Multiply the fluxion of each quantity into the product of all the rest, and the sum of all the products is the fluxion required.

Examples.

Ex. 1. The fluxion of x^2y^3 is $x^2\times 3y^2\dot{y}+y^3\times 2x\dot{x}$ $=3x^2y^2\dot{y}+2y^3x\dot{x}$.

Ex. 2. The fluxion of $y^{\frac{7}{2}}x^{\frac{5}{3}}z$ is $x^{\frac{5}{3}}z\times\frac{7}{2}y^{\frac{5}{2}}\dot{y}+y^{\frac{7}{2}}z\times\frac{5}{3}x^{\frac{2}{3}}\dot{x}+y^{\frac{7}{2}}x^{\frac{5}{3}}\dot{z}=\frac{7}{2}x^{\frac{5}{3}}zy^{\frac{5}{2}}\dot{y}+\frac{5}{3}y^{\frac{7}{2}}zx^{\frac{2}{3}}\dot{x}+y^{\frac{7}{2}}x^{\frac{5}{3}}\dot{z}$.

Ex. 3. The fluxion of $w^mx^ny^rz^s$ is $nx^ny^rz^sw^{m-1}\dot{w}+mw^my^rz^sx^{n-1}\dot{x}+rw^mx^nz^sy^{r-1}\dot{y}+sw^mx^ny^rz^{s-1}\dot{z}$.

Ex. 4. To find the fluxion of $x^2\times\overline{a^4+y^4}^{\frac{3}{2}}$.

By the last rule, the fluxion of $\overline{a^4+y^4}^{\frac{3}{2}}$ is $\frac{3}{2}\times\overline{a^4+y^4}^{\frac{1}{2}}\times4y^3\dot{y}=6\times\overline{a^4+y^4}^{\frac{1}{2}}\times y^3\dot{y}$; hence, the fluxion required is $x^2\times6\times\overline{a^4+y^4}^{\frac{1}{2}}\times y^3\dot{y}+\overline{a^4+y^4}^{\frac{3}{2}}\times2x\dot{x}$.

Ex. 5. To find the fluxion of $\sqrt{a^2+x^2}\times\sqrt{b^2+y^2}$. Find the fluxion of each part by the last rule, and the fluxion required is $\sqrt{a^2+x^2}\times\frac{\dot{y}y}{\sqrt{b^2+y^2}}+\sqrt{b^2+y^2}\times\frac{\dot{x}x}{\sqrt{a^2+x^2}}$.

¶ 16. It appears from this Prop. that the fluxion of xy consists of two parts, $\dot{x}y$ and $\dot{y}x$, the former part arising from the increase of y by $\dot{y}$, and the latter from the increase of x by $\dot{x}$; but if x should decrease whilst y increases, then the fluxion, expressing the *increase* of xy upon the whole, will be $\dot{x}y-\dot{y}x$, being the increase *minus* the decrease. Hence, to express the rate at which any quantity *increases*, the fluxion of the parts which increase must be written with the sign +, and those which decrease with the sign −. Now the increasing quantity is considered as positive; but if a negative quantity increase in magnitude, it must be considered as a decreasing quantity, and its fluxion will be negative. In like manner, a negative quantity decreasing in magnitude must be considered as an increasing quantity, and its fluxion will be positive. If therefore the fluxions of increasing quantities be writ-

ten with the sign +, and of decreasing with −, whenever the fluxion of any quantity is positive, it shows that quantity to be in an increasing state; and when negative, to be in a decreasing state.

Prop. VIII.

To find the fluxion of a fraction $\frac{x}{y}$.

¶ 17. Put $z=\frac{x}{y}$, then $zy=x$, and $\dot{z}y+y\dot{z}=\dot{x}$ (¶ 14.); $\therefore \dot{z}=\frac{\dot{x}-z\dot{y}}{y}=\frac{\dot{x}-\frac{x}{y}\times\dot{y}}{y}=\frac{y\dot{x}-x\dot{y}}{y^2}$. Hence, we find the fluxion of a fraction by the following

RULE:

From the fluxion of the numerator multiplied into the denominator, subtract the fluxion of the denominator multiplied into the numerator, and divide by the square of the denominator.

Examples.

Ex. 1. The fluxion of $\frac{x^2}{y^3}$ is $\frac{2y^3x\dot{x}-3x^2y^2\dot{y}}{y^6}=\frac{2yx\dot{x}-3x^2\dot{y}}{y^4}$.

Ex. 2. The flux. of $\frac{x+y}{z^3}$ is $\frac{z^3\times\overline{\dot{x}+\dot{y}}-\overline{x+y}\times 3z^2\dot{z}}{z^6}=\frac{z\times\overline{\dot{x}+\dot{y}}-\overline{x+y}\times 3\dot{z}}{z^4}$.

Ex. 3. The flux. of $\frac{xy}{z^2}$ is $\frac{z^2\times\overline{x\dot{y}+y\dot{x}}-xy\times 2z\dot{z}}{z^4}=\frac{z\times\overline{x\dot{y}+y\dot{x}}-2xy\dot{z}}{z^3}$.

Ex. 4. The fluxion of $\frac{a}{x}$ is $\frac{-a\dot{x}}{x^2}$; for a being constant, the fluxion of the numerator is nothing, and therefore the fluxion of the numerator multiplied into the denominator is nothing; in this case therefore, the fluxion of the fraction is *minus* the fluxion of the denominator multiplied into the numerator, divided by the square of the denominator.

Ex. 5. The fluxion of $\frac{1}{x^n}$ is $\frac{-nx^{n-1}\dot{x}}{x^{2n}} = -\frac{n\dot{x}}{x^{n+1}} = -nx^{-n-1}\dot{x}$; or the fluxion of $x^{-n} = -nx^{-n-1}\dot{x}$; when therefore the index of a quantity is negative, the fluxion is found by the same rule (¶ 13.) as when the index is positive.

Ex. 6. The fluxion of $\frac{\sqrt{a^2+x^2}}{\sqrt{b^2+y^2}}$ is

$$\frac{\sqrt{a^2+x^2}^{-\frac{1}{2}} \times x\dot{x} \times \sqrt{b^2+y^2} - \sqrt{b^2+y^2}^{-\frac{1}{2}} \times y\dot{y} \times \sqrt{a^2+x^2}}{b^2+y^2}$$

$$= \frac{x\dot{x}}{\sqrt{a^2+x^2} \times \sqrt{b^2+y^2}} - \frac{\sqrt{a^2+x^2} \times y\dot{y}}{\sqrt{b^2+y^2}^{\frac{3}{2}}}.$$

The putting of a quantity into fluxion is called the *direct* method of fluxions.

SCHOLIUM.

¶ 18. In questions of a geometrical and philosophical nature, where we want to get the relation of the fluents from the fluxions, and in others where we want to find whether quantities are positive or negative from the relation of them to their fluxions, it is necessary to pay regard to the signs of the fluxions, as explained in ¶ 16. But in putting equations into fluxions, as in the Problems de Maximis et Minimis, although one variable quantity may increase at the same time that another decreases, yet we may write the fluxion of each positive; for by writing it so in each equation in order to obtain the same fluxion from the different equations, the result will not be altered. In these, and such like cases, we may therefore make the fluxion of each quantity positive. We may further observe, that when any fluxion becomes negative according to the above rule, the quantity which expresses its value becomes negative. For instance,

if r=the radius of a circle, x=the versed sine, y=the right sine of an arc, then $y^2=2rx-x^2$, and $\dot{y}=\frac{r\dot{x}-x\dot{x}}{y}$; now for the first quadrant, x and y increase, and each fluxion is positive, and the value of $\dot{y}$ is positive, x being less than r; but in the second quadrant, y decreases and its fluxion becomes negative, and its value becomes negative, x being greater than r. This circumstance is similar to the case of a quantity passing through 0 and changing its sign, for $\dot{y}=0$ at the end of the quadrant.

¶ 19. When we compare the fluxions of two quantities, by comparing the increments that would be *uniformly* generated in a given time, the quantities have been supposed to be homogeneous, there being no relation between those which are not homogeneous; yet if, of two heterogeneous quantities, the *numerical value* of one be expressed in terms of the other, it is manifest that there will be no impropriety in expressing the fluxion of one in terms of the fluxion of the other. If one side of a right angled parallelogram be represented by 6 and the other by 9, we say, $6\times 9=54$ the area; our numerical operation is perfectly correct, but no one ever imagined that the units represented by 54 are homogeneous to the units represented by 6 and 9; if 6 and 9 represent inches in *length*, 54 will represent so many *square* inches, or so many *square* areas, the side of each of which is 1 inch in length. Or if a and x represent the two sides, the area of the parallelogram will actually be ax, referring that quantity to its proper units; although therefore there is no relation between the area and either of its sides, yet it is expressed in terms of the sides. And if a be constant and x variable, the fluxion of the area will be $a\dot{x}$ by Prop. 3; if therefore ($\dot{x}$) the fluxion of the abscissa x be 1 inch in *length*, the corresponding fluxion of the area will be a square inches; if $\dot{x}$ be 2 inches in *length*, the fluxion of the area will be $2a$ square inches. And in general, when we consider any two quantities which are not homogeneous, although their fluxions, which are expressed by their increments *uniformly* generated in a given time, can have no relation to each other, if we carry our ideas no further than the increments themselves; yet when we consider the numerical values of these fluxions, the analytical expression for one may be comprised in terms of the other without any impropriety, and our conclusions will be perfectly just and correct, in the sense in which the units of the respective quantities are understood, notwithstanding the fluxions themselves may be heterogeneous. Sir I. Newton, in his *Quadrature of Curves*, in finding the area of a curve, describes a parallelogram on the abscissa (x), the other side (a) of which is constant; and then he compares the

fluxion of the area of this parallelogram with the fluxion of the area of the curve, they being homogeneous quantities; and the fluxion of the area of the parallelogram being $\dot{ax}$, he gets the fluxion of the area of the curve. From what has been said above, when we reduce these matters to calculation, there appears to be no absolute necessity for this; but it is more scientific to make the comparison between homogeneous quantities, than between those which are not homogeneous, and therefore the former method is always to be preferred in cases where it can be applied, notwithstanding the conclusions which are otherwise deduced are perfectly true and satisfactory.

¶ 20. The ingenious and justly celebrated author of the *Analyst* has endeavoured to show, that the principles of fluxions, as delivered by its author, are not founded upon reasoning strictly logical and conclusive. He lays this down as a Lemma: "If you make any supposition, and in virtue thereof deduce any consequence; if you destroy that supposition, every consequence before deduced must be destroyed and rejected, so as from thenceforward to be no more supplied or applied in the demonstration." This, he thinks, is so plain as to need no proof. It may perhaps be admitted to be true, when we want to deduce the *absolute* value of a quantity which is to be obtained in virtue of a supposition; but it is not true when we want to obtain the *relative* values of quantities. He seems not to have attended to the connection which there must necessarily be between the two terms which constitute a ratio, and the two terms which express the ratio to which they approach as their limit, when you diminish them sine limite, called the *limit* of the ratio. It is agreed, that by diminishing the increments you approach to the ratio of the velocities which they had at the points from whence they began to be generated, and that by making them become indefinitely small, you arrive at a ratio indefinitely near to that of the velocities at those points. Now the two terms expressing the *limit* of a ratio must depend upon the terms themselves of the ratio, and the terms are obtained upon the supposition of the existence of an increment; the *limit* therefore is obtained upon the supposition of the existence of an increment; but the *limit* is a certain determinate invariable ratio, totally independent of the *magnitude* of the terms of the ratio, and consequently of the increment, as appears by ¶ 8. When we therefore deduce the *limit* by making the increments vanish, the *effect* of the prior existence of the terms of the ratio still remains in the terms which express the *limit* of the ratio. If the *existence* of the *terms* which express the *limit* of the ratio, depended upon the *existence* of the *terms* themselves of the ratio, the supposition which makes the latter vanish would necessarily make the former also vanish, and then no conclusion could be deduced by making the terms of the ratio vanish;

but as that is not the case, the *limit*, which is obtained by making the terms become equal to nothing, contains an effect, after the increments are actually vanished, which depends upon their having existed. The lemma therefore of the author, however true it may be under some circumstances, cannot be applied against the reasoning upon which the Principles of Fluxions are founded. The author admits the conclusions to be true. He says, "I have no controversy about your conclusions, but only about your logic; and it must be remembered that I am not concerned about the truth of your theorems, but only about the way of coming at them." The above observations show, not only that our conclusions are true, but that they are deduced by steps which are perfectly satisfactory, and strictly logical.

ARTICLE XXI.

New Questions to be solved in the next Number.

I. Quest. 62. *By A. Rabbit, at Uncle Sam's, Downbelow.*

Edward Shepherd, from English Yorkshire, has in page 198 of his Columbian Accountant, recorded and answered the following question.

"A, B, and C are to share £100000 in the proportion of $\frac{1}{3}$, $\frac{1}{4}$ and $\frac{1}{5}$ respectively; but C's part being "lost by his death, it is required to divide the *whole* "*sum properly* between the other two." This eminent author makes A's share=£$57142\frac{2}{3}\frac{8}{2}\frac{2}{9}$, and B's = £$42857\frac{4}{3}\frac{7}{2}\frac{7}{9}$. Now I demand the operation, by which we ought to find the shares of A and B; and also by what curious process Shepherd created the fractional tails of his answers.

II. Quest. 63. *By G. Baron, New-York.*

Thirty political sages meet every evening, at T. H. Kennedy's Porter-House in New-York. They consist of Royalists, Federalists, Clintonians, and Lewisites, in an increasing continual proportion; and their third difference is 2. I demand the number of each.

III. Quest. 64. *By Henry Smith, New-York.*

At what times of the day do the hour and minute hands of a watch form one continued straight line?

IV. Quest. 65. *By William Lenhart, Baltimore.*

Prove that $a+b$ has an infinite number of algebraic divisors.

V. QUEST. 66. *By Thomas Whittaker, York-Town, Penn.*

It is required to divide the arc of a semicircle into two arcs, so that the radius of the semicircle may be a mean proportional between the chords of those arcs.

VI. QUEST. 67. *By Peter Spangler, York-Town, Penn.*

If a trapezium be described about a circle, the sum of any two of its opposite sides is equal to the sum of the other two sides; required a demonstration.

VII. QUEST. 68. *By Joseph Cowing, Alexandria.*

The base of a plane rectilineal triangle being 14, its area equal to double that of its inscribed square, and one of the angles at the base 53° 03′; quere the other two sides of the triangle?

VIII. QUEST. 69. *By William Lenhart, Baltimore.*

Given the area of a plane rectilineal triangle=840; the length of a straight line drawn from the vertical angle to the middle of the base=39, and the vertical angle=61° 56′: to determine the triangle.

IX QUEST. 70. *By John D. Craig, Philadelphia.*

Describe a circle that shall touch the arc of the quadrant of a given circle, one of its bounding radii, and a semicircle described on the other bounding radius.

X. QUEST. 71. *By David Sanford, Newtown, Conn.*

Given the base and vertical angle of a plane rectilineal triangle, and the distance between the centres of its circumscribing and inscribing circles; to determine the triangle.

XI. PRIZE QUEST. 72. *By Joseph Cowing, Alexandria.*

Let a point be assumed in the circumference of a given circle; it is required to find, geometrically, another point in a straight line, passing through the assumed point and the centre of the circle, from which if a tangent be drawn, and a straight line drawn from the assumed point to the point of contact, this last shall obtain a given ratio to the straight line joining the assumed point and that which is required.

ARTICLE XXII.

Solutions of the Questions proposed in Article XXI.

I. Quest. 62. *Solved by A. Rabbit, at Uncle Sam's, Downbelow.*

Because C's *part* was lost by his death, it is plain that $\frac{1}{3} : \frac{1}{4} :: 4 : 3 ::$ the share of A : the share of B.

Hence $7 : 4 :: £100000 : £57142\frac{6}{7}$ = A's share;
and $7 : 3 :: £100000 : £42857\frac{1}{7}$ = B's share.

In answering this question, Shepherd first divided the given sum between A, B, and the ghost of C, in the ratios of $\frac{1}{3}$, $\frac{1}{4}$ and $\frac{1}{5}$; and afterwards divided the share of the ghost between A and B, in the ratio of $\frac{1}{3}$ to $\frac{1}{4}$. This is the curious process, by which Shepherd created the fractional tails of his answers.

II. Quest. 63. *Solved by Enoch Lewis, Westown, Pennsylvania.*

Let the required number of Royalists, Federalists, Clintonians, and Lewisites, be respectively represented by x, xy, xy^2 and xy^3; then by the question and the nature of differentials, we have

$$(y^3+y^2+y+1)\times x = 30,$$

and $(y^3-3y^2+3y-1)\times x = 2$. From these two equations we have

$y^3-3y^2+3y-1 : y^3+y^2+y+1 :: 1 : 15$; or $7y^3-23y^2+22y=8$; which resolved, gives $y=2$. Hence is easily found $x=2$, $xy=4$, $xy^2=8$, and $xy^3=16$: consequently our 30 political sages consist of 2 Royalists, 4 Federalists, 8 Clintonians, and 16 Lewisites.

III. QUEST. 64. *Solved by the proposer, Henry Smith, New-York.*

It is plain that the hour and minute hands of a watch form one continued straight line at the hour of six. And, since the lengths of the two hands may be considered as equal; if the length of the arc described by the hour hand in one hour, be put $=1$; the length of that described by the minute hand, in the same time, will be $=12$; and the difference of these two arcs will be $=11$. Now, when this difference becomes $=12$, the hands will again form a continued straight line: therefore by proportion 11 : 12 : : 1 *hour* : 1 *hour* $5\frac{5}{11}$ *min.* Further, since the angular velocities of the two hands are always invariable, the required times are evidently in arithmetic progression; and their common difference is 1 *hour*, $5\frac{5}{11}$ *min.*

Consequently at 6^h, $7^h\ 5\frac{5}{11}{}^m$, $8^h\ 10\frac{10}{11}{}^m$, $9^h\ 16\frac{4}{11}{}^m$, $10^h\ 21\frac{9}{11}{}^m$, $11^h\ 27\frac{3}{11}{}^m$, $0^h\ 32\frac{8}{11}{}^m$, $1^h\ 38\frac{2}{11}{}^m$, $2^h\ 43\frac{7}{11}{}^m$, $3^h\ 49\frac{1}{11}{}^m$; and $4^h\ 54\frac{6}{11}{}^m$, the hour and minute hands of a watch are diametrically opposite to each other, or form one continued straight line.

IV. QUEST. 65. *Solved by Enoch Lewis.*

It is evident that $\frac{a}{m}+\frac{b}{m}$ is an algebraic divisor of $a+b$; assume $m=2$, 3, 4, &c. and the expressions are still algebraical; whence, as the assumed value of m may be infinitely varied, the number of algebraic divisors of $a+b$, is infinite.

The same, solved by John Capp, Harrisburg, Pennsylvania.

It is easily proved, that when n denotes any odd number whatever, $a+b$ is a divisor of a^n+b^n. Now, because $a=\overline{a^n}^{\frac{1}{n}}$, and $b=\overline{b^n}^{\frac{1}{n}}$, it follows that $a^{\frac{1}{n}}+b^{\frac{1}{n}}$ is a divisor of $a+b$. Hence since n is any odd number, and an infinity of odd numbers exists, $a+b$ has an infinite number of algebraic divisors.

V. QUEST. 66. *Solved by Jacob Winooski, among the Green Mountains, Vermont.*

This question is only a particular case of the following general

PROBLEM.

In a given arc of a circle, to find a point from which a straight line drawn to the middle of the chord of that arc, shall be a mean proportional between two straight lines drawn from the required point to the extremities of the same chord.

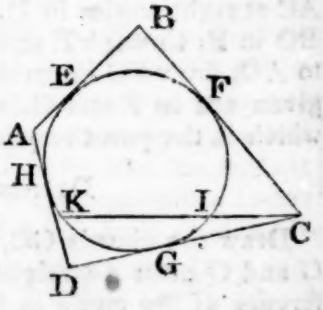
Let ABC be the given arc of a circle ABD; and AC its chord. With the straight line BD bisect AC at right angles in O. Bisect BO in E, through E and parallel to AC, draw FG intersecting the given arc in F and G, either of which is the point required.

DEMONSTRATION.

Draw the chords GC, GA and GB; and through G and O draw a straight line to intersect the circumference of the circle in H. Make the arc GI=BG; and draw the chords IG and IC. By construction, and the cor. to prop. 1. book 3. Sim. Euc. BD pas-

ses through the centre of the circle; and hence by prop. 30. book 3. Euc. the arc AFB=BGC. Again, BD passing through the centre of the circle, cuts AC and its parallel FG at right angles; therefore by prop. 3. book 3. Euc. FE=EG, and by prop. 30. book 3. Euc. the arc FB=BG, by construction=GI. But the arc AFB=BGC, consequently the arc AF=GC. In the triangles BGE and EGO, right angled at E, the side BE=EO by construction, and EG is common; therefore by prop. 4. book 1. Euc. BG=GO, and the angle BGF=FGH; whence by prop. 26. book 3. Euc. the arc FH=FB, by what has been shown,=GI. But it has been shown that the arc AF=GC, consequently the arc AH=CI; and by prop. 27. book. 3. Euc. the angle AGH=IGC. Further by prop. 21. book 3. Euc. the angle GAC=CIG. From what has been already demonstrated, it follows that the triangles AGO and GCI are similar, and hence AG : GO : : GI : GC. But by construction, the arc GI=GB, and therefore by prop. 29. book 3. Euc. the chord GI=GB, which has been proved=GO; consequently AG : GO : : GO : GC, as was to be demonstrated.

VI. Quest. 67. *Solved by James M'Ginness, Harrisburg, Pennsylvania.*



Let ABCD be the trapezium, touching the circle EFGH in the points E, F, G and H. From C draw a straight line cutting the circumference of the circle in the points I and K. Then by prop. 36. book 3. Euc. CI.CK $=\overline{CG}^2$, and also $=\overline{CF}^2$; consequently CF=CG. In a similar manner it may be demonstrated that FB=BE, AH=AE, and DH

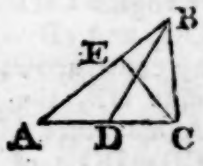
$=DG$. Whence $CF+BF+AH+HD=CB+AD=CG+BE+AE+DG=AB+DC$, as was to be demonstrated.

Corollary. $BC \sim AB=DC \sim AD$, and $BC \sim DC=AB \sim AD$.

VII. QUEST. 68. *Solved by John Capp.*

Let $b=14$ the given base, and $x=$the altitude, or perpendicular height of the triangle. Then it is well known that $\frac{bx}{b+x}=$ the side of the inscribed square; that $\frac{1}{2}bx=$the area of the triangle; and that $\frac{b^2x^2}{b^2+2bx+x^2}=$ the area of the inscribed square. Hence by the question $\frac{b^2x^2}{b^2+2bx+x^2}=\frac{bx}{4}$, or $b^2-2bx+x^2=0$: consequently $x=b=14$; and by plain trigonometry, the two required sides of the triangle are easily found to be 17·5184 and 14·4235.

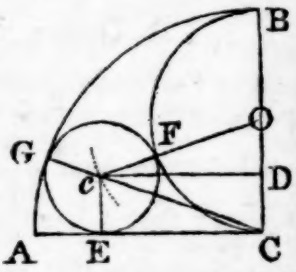
VIII. QUEST. 69. *Solved by William Lenhart, Baltimore.*



Let ABC be a plane rectilineal triangle, in which are given the area$=a=840$, D the point in the middle of the base AC, the length of the straight line $BD=b=39$, and the vertical angle $ABC=61^\circ\ 56'$. Draw CE perpendicular to AB, and put $t=$·5332029 the natural cotangent of ABC, the radius being$=1$; then by trigonometry CE : BE : : 1 : t, or BA.CE : BA.BE : : 1 : t. But by mensuration $BA.CE=2a$; and by Quest. 46. Math. Corr. $BA.BE=\overline{BD}^2-\overline{DC}^2=b^2-\overline{DC}^2$. Whence by substitution $2a : b^2-\overline{DC}^2$: : 1 : t; and by reduction $DC=\sqrt{b^2-2at}=$

25·00438 : of consequence AC and the other two sides, and angles of the triangle are easily found.

IX. QUEST. 70. *Solved by John D. Craig, Baltimore.*

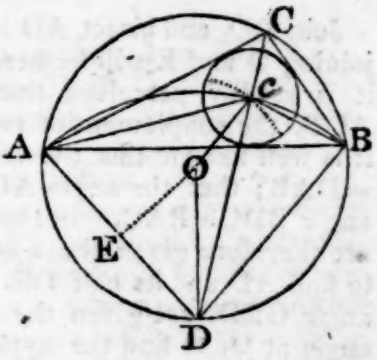


Let ABC be a quadrant of a given circle, C its centre, and CA and CB its bounding radii. Let O be the centre of a semicircle BFC described on CB as a diameter. Bisect OC in D; through D, and parallel to AC, draw D*c*; on the centre O, with a radius=BD, describe an arc cutting D*c* in *c*; from *c* draw the straight line *c*E perpendicular to AC; and on the centre *c*, with the radius *c*E, describe EGF, the required circle.

DEMONSTRATION.

Draw the straight line O*c*, intersecting the semicircle BFC in F; also draw the straight line C*c*G, meeting the arc BGA in G: then by construction, E*c*DC is a rectangle, and *c*E=CD=DO. In the triangles C*c*D and *c*DO, right angled at D, CD=DO, and *c*D is common; and hence *c*C=*c*O,=BD by construction. But since the radius OF=the radius OB, the remainder *c*F=the remainder DO. Again, since CG=CB, and *c*C=BD, therefore *c*G=CD=DO: of consequence *c*E=*c*F=*c*G. Further, it is easy to perceive that the circles BFC and GFE touch each other in F, a point in the straight line O*c*, joining their centres; and for the same reason, the circles BGA and GFE touch each other in the point G. Lastly, because the radius *c*E is perpendicular to AC, the circle GFE touches AC in the point E. Hence the circle GFE has been described as was required in the question.

X. QUEST. 71. *Solved by David Sanford, Newtown, Connecticut.*



On AB the given base, by prop. 33. book 3. Euc. describe a circle ACBD, such that the segment ACB, may contain an angle, equal to the given vertical angle of the required triangle. Bisect the arc ADB in D; join DA and DB; on the centre D, with the radius DA =DB, describe the arc A*c*B; and on O, the centre of the circle ACBD, with a radius=the given distance of the circumscribed and inscribed circles, describe an arc, intersecting A*c*B in *c*. Through the points D and *c*, draw the straight line D*c*C, meeting ACB in C; join AC and CB; and ACB is the triangle required.

DEMONSTRATION.

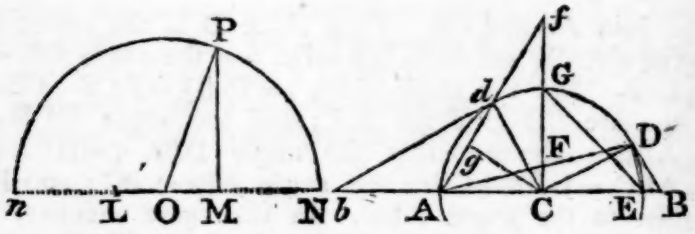
Join A*c* and *c*B. Then since D is the centre of the circular arc A*c*B, DB=D*c*, and the angle DB*c*= D*c*B. But the angle DB*c*, (=D*c*B),=DCB+CB*c*; the arc BD=the arc DA; and the angle DCB= ABD. Consequently the angle DB*c*, (=DBA+ AB*c*),=DBA+CB*c*; the angle AB*c*=*c*BC; and B*c* bisects the angle ABC. In the same manner, it may be proved that A*c* bisects the angle CAB; and hence, by prop. 4. book 4. Euc. *c* is the centre of the circle inscribed in the triangle ACB. Lastly, by construction, AB=the given base; ACB=the given vertical angle; and O*c*=the given distance between the centres of the circumscribed and inscribed circles:

therefore ACB is the triangle sought, as was to be demonstrated.

METHOD OF CALCULATION.

Join DO, and bisect AD in E; then a straight line joining O and E will be perpendicular to AD. Now it is readily perceived that DO bisects the angle ADB, the supplement of the given angle ACB; and it is well known that the angle ACD=DCB=ABD =DAB; that the angle ADC=ABC; and that the angle BDC=BAC. In the isosceles triangle ADB, are therefore given the side AB and all the angles, to find AD and its half DE. In the right angled triangle OED, are given the side DE and the oblique angle at D, to find the hypotenuse DO. In the triangle cDO, are given DO, cO and Dc=AD, to find the angle ODc: whence the angles ADC and CDB, respectively equal to the sum and difference of ADO and ODC, will also be known. Lastly, in the triangle ACB, are given the base AB and all the angles, to find the two unknown sides AC and CB.

XI. PRIZE QUEST. 72. *Solved by the proposer, Joseph Cowing, Alexandria.*



Let C be the centre, AE the diameter of the given circle AGE &c. and A the assumed point. Also let the given ratio be that of LM to CE. From the centre C, and at right angles to AE, draw CG, bisecting the arc of the semicircle AGE in G; and join

GE. From the point M, and perpendicular to LM, draw MP=GE; bisect LM in O; join OP; and on the centre O, with the radius OP, describe a semicircle, cutting LM produced in N and *n*. In the given circle AGE &c. inscribe AD=*n*M and A*d*=MN; from D and *d* draw the tangents DB and *db*, meeting AE produced in B and *b*: then either B or *b* is the point required.

Demonstration.

Draw AD and A*d*, intersecting CG in F and *f*; and on A*d* let fall the perpendicular C*g*. Join DE, *d*A, CD and C*d*. Then by similar triangles AD : AB : : DF : DC=CE; and A*d* : A*b* : : *df* : *d*C=CE. Also AD.AF=AE.AC=2.$\overline{CE}^2$=$\overline{GE}^2$=$\overline{MP}^2$, =*n*M.MN by construction; and A*g*.A*f*=$\overline{AC}^2$=$\overline{CE}^2$, or (2.A*g*.A*f*=) A*d*.A*f*=2.$\overline{CE}^2$=*n*M.MN. But AD was made=*n*M, and A*d*=MN; therefore AF=MN, and A*f*=*n*M. Whence DF=*df*=LM; and consequently AD : AB : : LM : CE, and A*d* : A*b* : : LM : CE, as was to be demonstrated.

Scholium.

The point B can be found only when LM is less than CE; and in this case the arc DE is always less than a quadrant. The point *b* may always be found; and the arc A*d* is universally less than a quadrant. All this may be easily demonstrated from the above construction.

ACKNOWLEDGEMENTS, &c.

A list of the names and places of residence of the ingenious gentlemen who have given true solutions to questions proposed in No. VII. The figures annexed to the names refer to the questions answered by each, as numbered in the work.

John Capp, Harrisburg, Pennsylvania,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11.
Jacob Cochran, Rockland, New-Jersey,	1, 3.
Joseph Cowing, Alexandria,	7, 11.
John Craggs, near Richmond, Virginia,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11.
John D. Craig, Baltimore,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11.
John Gaultier, New-Orleans,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11.
Martin Gibson, George-Town, South-Carolina,	1, 2, 3, 4, 5, 6, 7, 8, 9.
Rev. T. P. Irving, Newbern, North-Carolina,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11.
Enoch Lewis, Westown, Pennsylvania,	2, 3, 4, 5, 6, 7, 8, 9, 10, 11.
William Lenhart, Baltimore,	4, 8, 11.
James M'Ginness, Harrisburg, Pennsylvania,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11.
Warder Morris, Reading, Pennsylvania,	1, 2, 3, 4.
Thomas Maughan, Quebec,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11.
George Nagle, Reading, Pennsylvania,	1, 2, 3, 4.
George Quidor, Rockland, New-Jersey,	3.
Seth Robson, Savannah, Georgia,	1, 3, 4, 5, 6, 7, 9.
Charles Richards, Reading, Pennsylvania,	1, 2, 3, 4.
David Sanford, Newtown, Connecticut,	10.
Henry Smith, New-York,	1, 2, 3, 4, 5, 6, 7, 8, 9.
John Smithis, Philadelphia,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11.
Peter Spangler, York-Town, Pennsylvania,	6.
Jacob Winooski, Green Mountains, Vermont,	1, 3, 5, 6.
E. R. White, Danbury, Connecticut,	3, 11.
James Williams, Petersburg, Virginia,	1, 2, 3, 4, 5, 6, 7, 8, 9, 11.
John Wilson, Boston,	1, 3, 6, 7.
Diarius Yankee, Bunker's Hill,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11.
T. Youle, Portland, District of Maine,	1, 3.
Abner Young, Newhaven, Connecticut,	1, 3, 6, 7.
N. Young, near Lexington, Kentucky,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11.

The two best solutions to the prize question were given by *William Lenhart* of Baltimore, and *Joseph Cowing* of Alexandria. We could not agree on the merits of the two solutions; and therefore have awarded to each of those ingenious gentlemen a silver medal, value six dollars. Mr. Lenhart's medal is ready for delivery, and Mr. Cowing may receive his on the first day of April next. The solution published was determined by ballot.

The *health* of Mr. Baron, our principal editor, was last summer entirely destroyed, by three of the understrappers of the *Health Committee*. Deprived of his assistance, we earnestly solicit our contributors to endeavour to render their solutions as perfect as possible.

It is probable our first volume will be completed in ten and a half numbers. Any improvements in the solutions published, will be thankfully received, and inserted at the end of the volume.

Our country subscribers, who have already sent us one dollar, will please to remit us one dollar more; and those who have hitherto made no payments, must remit us two dollars, or their numbers will be stopped, and their names published.

ARTICLE XXIII.

The two first Chapters of Baron Maseres' Dissertation on the use of the Negative Sign in Algebra, published with a view of drawing the attention of American Mathematicians to a careful perusal of the original work.

CHAP. I.

The Definition of a Negative Quantity, in the sense in which it is used throughout this Dissertation; together with the plan or design of chap. 1 *and* 2.

¶ 1. The clearest idea that can, as I apprehend, be formed of a negative quantity, is that of a quantity that is subtracted from another greater than itself. To denote this subtraction, the subtracted quantity has the mark or sign—, which is called *minus*, prefixed to it, as the sign+, which is called *plus*, is prefixed to a quantity that is to be added to another. Thus if b is to be added to a (in which case a may be either greater or less than b), their sum will be wrote thus, $a+b$; if b is to be subtracted from a (in which case a must be greater than b), the remainder, or difference, will be wrote thus, $a-b$.

¶ 2. Hence it is evident that a single quantity can never be marked with either of these signs, or considered as either affirmative or negative; for if any

single quantity, as b, is marked either with the sign + or the sign − without assigning some other quantity, as a, to which it is to be added, or from which it is to be subtracted, the mark will have no meaning or signification; thus, if it be said that the square of −5, or the product of −5 into −5, is equal to +25, such an assertion must either signify no more than that 5 times 5 is equal to 25, without any regard to the signs, or it must be mere nonsense and unintelligible jargon: I speak according to the foregoing definition, by which the affirmativeness or negativeness of any quantity implies a relation to another quantity of the same kind to which it is added, or from which it is subtracted; for it may perhaps be very clear and intelligible to those who have formed to themselves some other idea of affirmative and negative quantities different from the above-defined.

¶ 3. Now, what I here propose, is to apply the operations of arithmetic to compound quantities, consisting partly of affimative and partly of negative quantities, or to the differences of simple quantities, and to demonstrate the rule usually given for multiplication, to wit, that − × + gives −, + × − gives −, + × + gives +, and − × − gives +, or that as often as the signs of the multiplier and multiplicand are alike, the product is to be marked +, and as often as they are unlike, the product is to be marked −.

CHAP. II.

Of the several Operations of Algebra.

SECT. I.

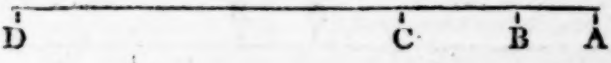
Of the Addition and Subtraction of Differences.

¶ 4. The addition of the differences of quantities is performed thus. If the difference of two quantities a and b, whereof a is the greater, is to be added to a

third quantity c, the greater quantity a must be added to c, and the lesser subtracted from the sum of c and a, and the remainder $c+a-b$ will be the sum of c and $a-b$ required; or in other words the compound quantity $a-b$ is added to c by connecting it, or writing it near to, or in the same line with c, without altering the signs of its members a and b.

This, if it be thought to want a proof, may be shown from the following principle, to wit, that "if from two unequal quantities, two equal quantities be subtracted, the difference of the quantities will not thereby be altered, but will be the same after the subtraction as before it;" for from hence 'tis evident, that if from each of the quantities $c+a$ and a the quantity b, which is less than either of them, be subtracted, the remainder of the former of these quantities will exceed the remainder of the latter by c, that is, the excess of $c+a$ above b exceeds the quantity $a-b$ by c; therefore the excess of $c+a$ above b, or the quantity $c+a-b$ is equal to the sum of c and $a-b$. Q. E. D.

FIGURE I.



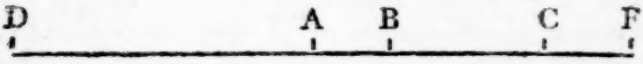
Thus if c is represented by the line DC, a by CA, b by BA, and (consequently) $a-b$ by CB, 'tis evident that if CA be added to DC, and BA be subtracted from the sum DA, the remainder DB will be equal to the line DC increased by CB, the difference of the lines CA, BA.

The same will be evident, if instead of $a-b$ we should add a compound quantity, consisting of any number of terms whatsoever, as $a-b+d-e+f$, or $a+d+f-b-e$, to c; to wit, that these quantities must be connected with c without altering their signs,

or that the sum of c and the compound quantity $a-b+d-e+f$ is equal to $c+a-b+d-e+f$. For let $a+d+f=m$, and $b+e=n$, and we shall have (by what has been already shown) the sum of c and $m-n$ equal to $c+m-n$; therefore the sum of c and $a+d+f-b-e$ is equal to $c+a+d+f-b-e$, or the sum of c and $a-b+d-e+f$ is equal to $c+a-b+d-e+f$.

¶ 5. The subtraction of the differences of quantities is performed thus. If the difference of two quantities a and b, whereof a is the greater, is to be subtracted from a third quantity c that is greater than that difference, the lesser quantity b must be added to c, and the greater quantity a must be subtracted from the sum of c and b; and the remainder $c+b-a$ will be equal to the excess of c above the compound quantity $a-b$ required. Or, in other words, the compound quantity $a-b$ is subtracted from c by connecting it, or writing it near to, or in the same line with c, and changing the signs of its members a and b.

FIGURE II.



The proof of this may be derived from the following principle, to wit, that "if to two unequal quantities, two equal quantities be added, their difference will not thereby be altered, but will remain the same after this addition as it was before it. For from hence it is evident, that if to each of the two unequal quantities c and $a-b$ the quantity b be added, the excess of $c+b$ above a, or the quantity $c+b-a$, will be equal to the excess of c above $a-b$. Q. E. D.

Thus if c is represented by the line DC, a by the line CA, b by BA, and (consequently) $a-b$ by CB,

'tis evident that if a line CF equal to BA be added to DC, and from the sum DF the line BF (which is equal to CA) be subtracted, the remainder DB will be equal to the excess of the line DC above CB the difference of the two lines CA, BA.

The same will be evident if instead of $a-b$ we should subtract a compound quantity, consisting of any number of terms whatsoever, as $a+d+f-b-e$, or $a-b+d-e+f$, from c, supposing c to be greater than that quantity, to wit, that these terms must be connected with c with their signs changed; or that the excess of c above the compound quantity $a-b+d-e+f$ is equal to $c-a+b-d+e-f$, or $c+b+e-a-d-f$. For let $a+d+f=m$, and $b+e=n$, and (by what has been already shown) we shall have the excess of c above $m-n$ equal to $c+n-m$; therefore the excess of c above $a+d+f-b-e$ is equal to $c+b+e-a-d-f$.

Sect. II.

Containing Demonstrations of two very important Principles of Arithmetic.

¶ 6. In order to demonstrate the rule for the multiplication of differences, or of compound quantities consisting partly of affirmative and partly of negative members, it will be necessary to premise the two following principles.

1st. If there be two quantities of any kind a and m, the product that arises by multiplying a by m is equal to the product that arises by multiplying m by a; or the product $m\,a$ is equal to the product $a\,m$. Thus, 3 times 7 is equal to 7 times 3.

2d. If any number of quantities a, b, c, are separately multiplied into any quantity m, the sum of the products $m\,a$, $m\,b$, $m\,c$, will be equal to the product that arises by first adding those quantities together,

and then multiplying their sum into m; or, to express the same thing in algebraic symbols, $\overline{a+b+c} \times m = m\,a + m\,b + m\,c$.

¶ 7. It may not perhaps be amiss, before we proceed any further, to make an observation or two concerning the meaning of the word multiplication, which is here supposed to extend to all kinds of quantities incommensurable as well as commensurable. The primary signification of this word is evidently repetition, or the taking a quantity over and over again a certain number of times, to wit, so many times as is expressed by the multiplicator, or as there are units in the multiplicator, or it is the finding a multiple of any proposed quantity; this is multiplying by a whole number: it afterwards came to be applied to fractions, and it became as common to speak of multiplying a quantity by a fraction as of multiplying it by a whole number; but this is in reality a double operation, consisting of a division and a multiplication; for it is multiplying an aliquot part of the multiplicand, or first dividing it into a certain number of equal parts, and then finding a multiple of one of those parts: thus multiplying a quantity by $\frac{4}{5}$ is first dividing it into 5 equal parts, and then finding a quantity that shall be equal to four of those parts. Now in both these senses it will be found to be true, that if we suppose the multiplicand and multiplicator to be both quantities of the same kind, for instance lines, their product will be a quantity of the same kind also, and will be a fourth proportional to the quantity or line called unit (or the quantity to which the multiplicator and multiplicand are referred, or whereof they are considered as either multiples or parts) the multiplicator and the multiplicand. As therefore this property belongs to both these kinds of multiplication, but is itself more extensive than either of them, and is capable of being applied to incommensurable as well as commensura-

ble quantities (proportionality being common to both those kinds of quantities), algebraists, when they found occasion to extend the operation, and name of the operation of multiplication, to incommensurable as well as commensurable quantities, or numbers, determined it by the fore-mentioned property, understanding always by the multiplication of two quantities the finding a fourth proportional to the quantity called unit, the multiplicator and the multiplicand; for all these quantities, and in general all the quantities that enter into the same calculation, are supposed to be quantities of the same kind.

¶ 8. Both the foregoing principles mentioned in ¶ 6, belong to common arithmetic; and the latter in particular is that upon which the common rules for multiplication are founded: thus, if any number, as 375, greater than those which are denoted by a single figure, is to be multiplied by 4, 'tis difficult to see at once what number 4 times 375 is equal to; therefore 'tis usual to divide it into three parts, to wit, 5, 70, and 300, and to multiply each of these parts by 4, and then add the products together, as follows: 4 times 5 is 20; 4 times 70 is 280, to which if we add 20 the sum will be 300; and 4 times 300 is 1200, to which if we add 300 (the sum of two former products) we shall have 1500 for the sum of the three products 20, 280, and 1200: and this sum is equal to the product of 375 into 4 by the latter of the foregoing principles. Now, though both these principles belong to common arithmetic (at least as far as they relate only to commensurable quantities, or numbers), and therefore might here be taken for granted, or be supposed to be already known, yet, as most writers on common arithmetic assume them without a demonstration, it may not be amiss to digress a little from the subject we are now upon, to give the following proofs of them.

¶ 9. The first of these principles may be proved

thus. The product $m\,a$ is a fourth proportional to 1, m, and a; that is, $1 : m :: a : m\,a$; therefore *permutando* $1 : a :: m : ma$. But the product $a\,m$ is a fourth proportional to 1, a, and m; that is, $1 : a :: m : a\,m$; therefore $m : m\,a :: m : a\,m$; consequently $m\,a$ is $= a\,m$. Q. E. D.

From this principle may be deduced the following corollaries, which are of the greatest use in algebraic and arithmetical computations.

Coroll. 1. If there be three quantities a, b, c, their product will be the same in whatever order they are multiplied. For since $a\,b$ is $= b\,a$, it follows that $a\,b \times c$ is $= b\,a \times c$, or $c\,a\,b = c\,b\,a$; in like manner, because $b\,c$ is $= c\,b$, $b\,c \times a$ will be $= c\,b \times a$, or $a\,b\,c = a\,c\,b$; and because $a\,c$ is $= c\,a$, $a\,c \times b$ will be $= c\,a \times b$, or $b\,a\,c = b\,c\,a$. But because $1 : b :: c : b\,c$, and $b : a\,b$ ($:: 1 : a$) $:: b\,c : a\,b\,c$, it follows (by El. 5, 22.) that $1 : a\,b :: c : a\,b\,c$. Again, because ($1 : c :: a : c\,a$, and *permutando*) $1 : a :: c : c\,a$, and $a : b\,a$ ($:: 1 : b$) $:: c\,a : b\,c\,a$, it follows (by El. 5, 22.) that $1 : b\,a :: c : b\,c\,a$, and consequently, because $b\,a$ is $= a\,b$, $1 : a\,b :: c : b\,c\,a$. But it was just now shown that $1 : a\,b :: c : a\,b\,c$; therefore $c : a\,b\,c :: c : b\,c\,a$, and consequently $a\,b\,c$ is $= b\,c\,a$; therefore $a\,c\,b = a\,b\,c = b\,c\,a = b\,a\,c$, or the four products $a\,c\,b$, $a\,b\,c$, $b\,c\,a$, $b\,a\,c$, are equal to each other; and in the same manner it may be shown that they are likewise equal to the two remaining products $c\,a\,b$ and $c\,b\,a$. Therefore all the six products that arise by multiplying together the quantities a, b, and c, according to all possible combinations, are equal to each other. Q. E. D.

Coroll. 2. In the same manner it may be shown, that if there be any four, or more, quantities, a, b, c, d, &c. the products that arise by multiplying them all together will be the same, in whatever order they are multiplied.

Coroll. 3. Hence it follows, that if m be the index of the powers of any two quantities a and b, $a^m \times b^m$ will be $=$

$\overline{ab}^m$, or the product of the mth powers of the two quantities will be equal to the mth power of the product of the quantities themselves. For, by Coroll. 2, $a \times a \times a \times a \times a \times$ &c. continued to m terms multiplied into $b \times b \times b \times b \times b \times$ &c. continued to the same number of terms, will be equal to $ab \times ab \times ab \times ab \times ab \times$ &c. continued likewise to m terms; that is, $a^m \times b^m = \overline{ab}^m$. Q. E. D.

This last corollary may be demonstrated in the following manner, from the well-known method of compounding, or adding ratios together, by multiplying antecedent into antecedent, and consequent into consequent; concerning which, see Dr. SANDERSON'S excellent discourse upon the composition and resolution of ratios, in the second volume of his Algebra. The ratio of $a^m \times b^m$ to 1 is equal to the sum of the ratios of a^m to 1 and of b^m to 1, and therefore to the sum of m times the ratio of a to 1 added to the sum of m times the ratio of b to 1, and therefore to m times the sum of the ratios of a to 1 and of b to 1, and consequently to m times the ratio of ab to 1, and therefore to the ratio of $\overline{ab}^m$ to 1; consequently $a^m \times b^m = \overline{ab}^m$. Q. E. D.

Coroll. 4. Hence it follows that $\sqrt[m]{a} \times \sqrt[m]{b} = \sqrt[m]{\overline{ab}}$. For put $\sqrt[m]{a} = d$, $\sqrt[m]{b} = e$, and $\sqrt[m]{\overline{ab}} = f$. Then will $d^m = a$, $e^m = b$, and $f^m = ab$; therefore $d^m e^m (= ab) = f^m$; and, by the last corollary, $de = f$, that is, $\sqrt[m]{a} \times \sqrt[m]{b} = \sqrt[m]{\overline{ab}}$. Q. E. D.

Coroll. 5. If a is $= \frac{c}{b}$, a^m will be $= \frac{c^m}{b^m}$. For since $a = \frac{c}{b}$, we shall have $ab = c$, and $a^m \times b^m =$ (by Coroll. 3.) $\overline{ab}^m = c^m$; consequently $a^m = \frac{c^m}{b^m}$. Q. E. D.

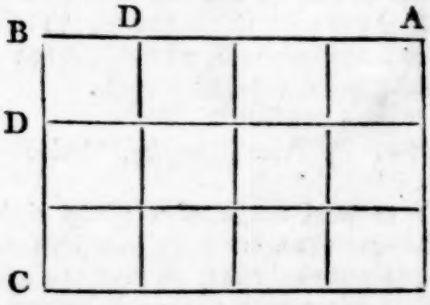
Coroll. 6. If a is $=\frac{c}{b}$, $\sqrt[m]{a}$ will be $=\frac{\sqrt[m]{c}}{\sqrt[m]{b}}$. For put $\sqrt[m]{a}=d$, $\sqrt[m]{b}=e$, and $\sqrt[m]{c}=f$; and we shall have $d^m=a$, $e^m=b$, and $f^m=c$, and consequently $d^m \left(=a=\frac{c}{b}\right)=\frac{f^m}{e^m}$. Therefore, by the converse of Coroll. 5, d will be $=\frac{f}{e}$, or $\sqrt[m]{a}$ will be $=\frac{\sqrt[m]{c}}{\sqrt[m]{b}}$. Q. E. D.

Coroll. 7. 'Tis evident that these corollaries are true of any number of quantities whatsoever, as well as of two quantities. Thus, $a^m\, b^m\, c^m\, d^m\, e^m$ &c. is $=\overline{a\,b\,c\,d\,e\,\&c.}^m$, and $\sqrt[m]{a}\,\sqrt[m]{b}\,\sqrt[m]{c}\,\sqrt[m]{d}\,\sqrt[m]{e}$ &c. $=\sqrt[m]{\overline{a\,b\,c\,d\,e\,\&c.}}$.

¶ 10. The second of these principles may be demonstrated in the case of whole numbers, from the analogy between arithmetical products and geometrical rectangles. For if it can be shown that whatever is demonstrated concerning the latter kind of quantity is applicable likewise to the former, the demonstration of the first proposition of the second book of EUCLID's Elements will be a demonstration of this second principle of arithmetic. Now this analogy is explained by Sir ISAAC NEWTON, in the beginning of his *Arithmetica Universalis*, in the following manner. If there be any two whole numbers a and b, and two right lines to be taken in the proportion of these whole numbers, the number of units in the arithmetical product of a and b will be equal to the number of geometrical squares of unity, or squares whose sides are equal to the line representing an unit, contained in the rectangle under the two lines. Thus if a be 3, and b be 4, and CB, BA, Fig. 3. be two lines in the proportion of three to four, so that CB shall represent the number 3, BA the number 4, and consequently BD, which is the third part of CB, or the fourth part of BA, shall represent a unit, the number of units contained in the nume-

rical product of 3 and 4 will be equal to the number of little squares, each of whose sides is equal to the line BD, contained in the rectangle AC; for in both the number is twelve: the reason of which is this; the rectangle may be considered as produced by adding together three rectangles, each equal to AD, whose base is the line AB, which represents the multiplicand 4, and its height BD, or unity, and which consequently contains 4 little squares of unity; and the numerical product of 4 into 3 is produced by taking the number 4 three times, or (to use the same expression as before) by adding together three numbers, each of which contains 4 units. And this reasoning, and consequently the analogy between products and rectangles derived from it, evidently holds of any other numbers whatsoever. Consequently, whatever is demonstrated concerning the proportions of geometrical rectangles (those proportions being the same with the proportions of the numbers of the squares of the line that represents an unit contained in them), is true likewise of the proportions of the products of numbers proportional to the sides of those rectangles. Thus, for instance, all the second book of EUCLID's Elements is true of products as well as rectangles.

FIGURE III.



¶ 11. But the proper proof of this principle in the case of whole numbers, or that which is purely alge-

braic, or derived immediately from the nature of quantity in general, without having recourse to the properties of any particular kind of quantity, as lines or surfaces, is that which EUCLID has given of it, in the first proposition of the fifth book of his Elements; he having there demonstrated, that, if there be two sets of quantities, a, b, c, and ma, mb, mc, whereof each quantity in the latter set contains each quantity in the former set m times, the whole latter set will contain the whole former set m times; that is, when expressed in algebraic characters, $ma+mb+mc$ is $= \overline{a+b+c} \times m$.

(To be continued.)

ARTICLE XXIV.

NEW QUESTIONS to be solved in the next Number.

I. QUEST. 73. *By Jacob Winooski, Green Mountains, Vermont.*

Five political vagabonds A, B, C, D and E, are to be transported from the city of New-York. The transportation of A, B, and C, will cost $ 101; of B, C, and D, $ 113; of C, D, and E, $ 116; of A, D, and E, $ 112; and of A, B, and E, $ 116: what will be the expence of transporting each.

II. QUEST. 74. *By Peter Spangler, York-Town, Penn.*

From the vertical angle of a given right angled plane rectilineal triangle, it is required to draw a straight line to cut the base, so that the area of the oblique angled triangle intercepted between the hypotenuse and the required line, the square of the base

of this oblique angled triangle, and the square of the required line, taken together, may be equal to the square of the hypotenuse.

III. QUEST. 75. *By James M'Ginness, Harrisburg, Pennsylvania.*

In page 142 of the Mathematical Correspondent, it is said that $\frac{\sqrt{3}}{2+\sqrt{3}} \times s = (2\sqrt{3}-3) \times s$; and in page 145 of the same work, it is asserted that $\frac{\sqrt{2}}{\sqrt{2}+1} \times s = (2-\sqrt{2}) \times s$: I demand a demonstration of the truth or falsity of these two equations.

IV. QUEST. 76. *By John Smithis, Philadelphia.*

In extracting the roots of the powers of numbers, why do we make the number of digits in a period, equal to the exponent of the power?

V. QUEST. 77. *By Thomas Maughan, Quebec.*

Required an investigation of the principles of Plain Sailing in Navigation.

VI. QUEST. 78. *By John Capp, Harrisburg, Penn.*

Determine, by algebra, the number of degrees, &c. contained in an angle, whose cosine is equal to its tangent.

VII. QUEST. 79. *By James M'Cormic, Carlisle, Penn.*

In a trapeziod are given one of the two parallel sides, considered as a base, the two angles adjacent to the base, and the area, to determine the altitude or perpendicular height of the figure.

VIII. QUEST. 80. *By the Rev. T. P. Irving.*

My tortoise-shell snuff-box, ingeniously wrought,
With iv'ry adorn'd and gold spangles,
Has sides, ends and bottom, all square like its top,
And equal its eight solid angles.
A ringlet, diag'nally, roll'd o'er its lid,
Completes an entire revolution,
And marks, by a punctum, a curve, that contains
The clew to this question's solution.
Three inches, and more, by two ones, nine and four,
Plac'd after the decimal sign, Sirs,
Extended on Gunter's old *iv'ry* or *box*,
The length of the curve, will define, Sirs.
The curve, tho', exceeds the diag'nal within ;—
Take out, then, by simple subtraction,
One, three nines, six, eight, nine, six, one and a five,
Express'd in a decimal fraction.
Moreover, this box cramm'd with choice Macuba,
One half day (the hours of vocation,)
Just serves to regale my olfactory nerves,
And stimulates sweet sternutation.
Pray find me a box, to regale me a day,
(Et erit extrem' obligatio ;)
Retaining the depth, make the length to the breadth
As fourteen to eight, in a ratio.

IX. QUEST. 81. *By Jacob Cochran, Rockland, New-Jersey.*

How far must a Ship sail on a S. W. by S. course, from a port in Lat. 38° 15′ N. before her Latitude and difference of Longitude become equal?

X. PRIZE QUEST. 82. *By John D. Craig, Baltimore.*

In a circular arc less than a quadrant, it is required to find the relation between that arc and its cotangent, when the rectangle of the arc and its cosine is a maximum.

ARTICLE XXV.

Solutions of the Questions proposed in Article xxiv.

I. Quest. 73. *Solved by the proposer, Jacob Winooski, Green Mountains, Vermont.*

Let the expenses of transporting A, B, C, D and E, be respectively represented by a, b, c, d, and e dollars. Then by the question,

$$a+b+c=101$$
$$b+c+d=113$$
$$c+d+e=116$$
$$a+d+e=112$$
$$a+b+e=116$$

Now the sum of these five equations, divided by 3, gives the equation F, or $a+b+c+d+e=186$; and the five original equations, severally subtracted from F leaves

G, or $d+e=85$
H, or $a+e=73$
I, or $a+b=70$
J, or $b+c=74$
K, or $c+d=70$.

Lastly it is evident that $F-\overline{J+G}=a+b+c+d+e-\overline{b+c+d+e}=a=186-159=27$; whence, from the equations I, J, K and H are found $b=43$, $c=31$, $d=39$, and $e=46$ dollars, the sums required.

II. Quest. 74. *Solved by the proposer, Peter Spangler, York-Town, Pennsylvania.*

Let ABC be the given right angled plane rectilineal triangle, and BD the required straight line; then by mensuration $\frac{1}{2}$.AD.BC=the area of the oblique angled triangle ABD. Now by the question $\frac{1}{2}$.AD.BC$+\overline{AD}^2+\overline{BD}^2=\overline{AB}^2$; and by prop. 12. book 2. Euc.

$2.AD.DC+\overline{AD}^2+\overline{BD}^2=\overline{AB}^2$: whence $2.AD.DC=\frac{1}{2}.AD.BC$, $2DC=\frac{1}{2}.BC$, and $DC=\frac{1}{4}.BC$.

Corollary. When the base is less than one-fourth of the perpendicular BC, the question is impossible.

III. QUEST. 75. *Solved by Henry Smith, New-York.*

It is well known that the product of the sum and difference of any two quantities is equal to the difference of their squares, and that the value of a fraction is not altered by an equal multiplication of its two terms.

Let now the numerator and denominator of $\frac{\sqrt{3}}{2+\sqrt{3}}\times s$, be each multiplied by $2-\sqrt{3}$, and we obtain $\frac{\sqrt{3}}{2+\sqrt{3}}\times s$ $=\frac{2\sqrt{3}-3}{1}\times s=(2\sqrt{3}-3)\times s$, the same is in page 142 of the Mathematical Correspondent. Again by multiplying the two fractional terms of $\frac{\sqrt{2}}{\sqrt{2}+1}\times s$, by $\sqrt{2}-1$ we have $\frac{\sqrt{2}}{\sqrt{2}+1}\times s=\frac{2-\sqrt{2}}{1}\times s=(2-\sqrt{2})\times s$, as was asserted in page 145 of the work above mentioned.

IV. QUEST. 76. *Solved by the proposer, John Smithis, Philadelphia.*

To answer this question I shall here demonstrate the following Lemma, which is in substance nearly the same as that in page 163 of the 7th edition of Bonnycastle's Arithmetic.

Lemma. The number of digits in the product of any two numbers can neither be greater than the sum of the digits in the two factors, nor less than the sum of the same digits diminished by a unit.

DEMONSTRATION.

9,99,999,9999, &c. are the greatest possible in respect to their number of digits. Now, the product of any two of these, suppose 999×99, is less than 999×100; but $999 \times 100 = 99900$, contains only as many digits are in 999 and 99: whence it evidently follows, that the number of digits in the product of any two numbers cannot exceed the sum of the digits in the two factors.

Again 1, 10, 100, 1000, &c. are the least possible with regard to their number of digits. And the product of any two of these, for instance $1000 \times 100 = 100000$, evidently contains a number of digits, which is one less than the sum of the digits in the two factors 1000 and 100. But since 1000 and 100 were the least possible in respect to their number of digits; the product of any two numbers respectively composed of the same number of digits will be greater than 100000.

Hence it is easy to infer that the number of digits in the product of any two numbers cannot be less than the sum of the digits in the two factors, diminished by a unit, as was to be demonstrated.

Corollary 1. The number of digits in any square number can neither be greater than twice the digits in the root, nor less than twice the digits of the same root, diminished by one.

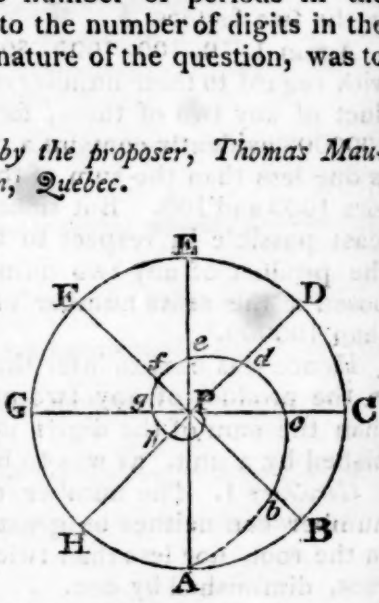
Cor. 2. The number of digits in any cube number can neither be greater than thrice the digits in the root, nor less than thrice the digits of the same root, diminished by two.

Cor. 3. Let a and n be any two numbers, and $m =$ the number of digits in a; then the number of digits in a^n can neither be greater than mn, nor less than $\overline{mn-n}-1$.

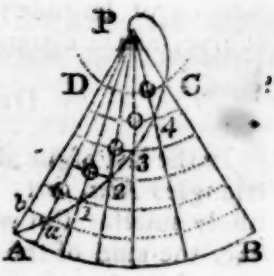
SOLUTION.

In extracting the roots of the powers of numbers, we begin at the units place, and separate the given power into periods, each containing a number of digits, equal to the exponent of the power. This well known process being attentively considered, in conjunction with the above lemma and corollaries, it will evidently appear that the number of periods in the power is always equal to the number of digits in the root; which, from the nature of the question, was to be shown.

V. QUEST. 77. *Solved by the proposer, Thomas Maughan, Quebec.*



In a projection of the spherical earth, let P be the pole, and PA, PB, PC, PD, &c. meridian circles, here represented by straight lines. Imagine a ship to sail from the point A on any oblique course PA*b*; and she will evidently intersect the meridians at a constant angle equal to that course. Hence her path will be a spiral line A*bcdef*, &c, making the angles PA*b*, P*bc*, P*cd*, P*de*, &c. all equal one to another, until the vessel arrive at the pole P. The line A*bcde*, &c. is by the theoretical writers on navigation called the loxodromic spiral, from the two Greek words, *loxos*, signifying *oblique*, and *dromos*, *a course*.



In the annexed projection of a portion of the spherical earth, P represents the pole, and PA and PB two meridians. Let a ship from A sail on an oblique course along the loxodromic spiral A*a*123, &c. until she arrive at C; through C and A draw the parallels of latitude CD and AB; and AD=CB is the *Difference of Latitude* between A and C. Further, since a ship's *Meridional Distance* is known to be "her "distance from the meridian she has left measured "on the parallel of latitude she is in;" it is plain that CD is the ship's meridional distance at C from A. But the *Meridional Distance* differs essentially from the *Departure*; for "*Departure* is the whole easting "or westing, continually made by a vessel, in steer-"ing an oblique course." To illustrate this, imagine the loxodromic distance AC to be divided into infinitely small equal parts, in the points *a*, 1, 2, 3, 4, &c. and through these points draw the meridians P*o*a, P*o*1, P*o*2, P*o*3, &c. and *ba*, *o*1, *o*2, *o*3, &c. arcs of parallels of latitude. Then since the meridians converge towards the pole P, the *Departure*, =*ba*+*o*1 +*o*2+*o*3+&c. is greater than DC the *Meridional Distance*. But had the ship sailed from C, on an equal and opposite course, until she arrived at A, her *Departure*, *o*C+*o*4+*o*3+*o*2+&c. would have been less than AB, her *Meridional Distance* at A from C. Having now shown the difference between Meridional Distance and Departure, I shall next demonstrate the following general

Theorem.

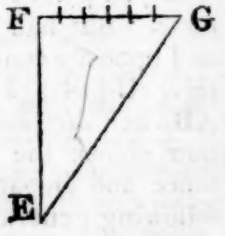
In sailing on an oblique course, radius is to the cosine of the course, as the distance sailed is to the dif-

ference of latitude; and radius is to the sine of the course, as the distance sailed is to the departure.

Demonstration.

In the last of the above two figures, the right angled triangles A*ba*, *ao*1, 1*o*2, &c. are by hypothesis infinitely small. Put radius=1, the cosine of the course =*c*, the sine of the course=*s*, and any variable portion of the loxodromic distance AC=*x*; then is the infinitely small distance A*a*=$\dot{x}$. But by Trigonometry $1 : c :: \dot{x} : c\dot{x}$ =A*b* the fluxion of the difference of latitude; $1 : s :: \dot{x} : s\dot{x}$=*ba* the fluxion of the departure. Now, since in every point of the loxodromic distance AC, the fluxion of the distance, that of the difference of latitude, and that of the departure, are as $\dot{x}$, $c\dot{x}$ and $s\dot{x}$; or as 1, *c*, and *s*; or as radius, cosine, and sine of the course; it follows that their contemporary fluents, the distance sailed, the difference of latitude, and the departure, are also in the same ratios; as was to be demonstrated.

Corollary 1. In sailing from A to C, on the oblique course P*A*a, let AD, the difference of latitude, went unbent from its circular curvature, be equal in length to the straight line EF, drawn in the direction of the meridian; and let the departure=*ba*+*o*1+*o*2+*o*3 +&c. when unbent from its circular curvatures, be equal in length to the straight line FG, drawn perpendicular to FE. The straight line GE, joining the points G and E, will be equal in length to the loxodromic distance AC; and the angle E, opposite to FG, will be equal to P*A*a, the course steered by the vessel. For, by Trigonometry, EG, EF and FG are as radius, cosine and sine of the angle E.



Corollary 2. The course, the distance, the difference of latitude, and the departure, may always be constructed by means of a plane rectilineal triangle, and calculated by Plain Trigonometry: and hence, say Mr. Emerson and other mathematicians of the greatest eminence, "originated the name of *Plain Sailing*."

SCHOLIUM.

Dr. Halley, Mr. Emerson, Dr. Hutton, and others, in perfect unison with all the mathematicians of modern times, have drawn their demonstrations of *Plain Sailing* from the principles here used. "But an ig- "norant rabble of quack navigators, has, as the "learned Mr. Hodgson, F. R. S. justly observes, poi- "soned the minds of seamen, with the pernicious "doctrine that Plain Sailing considers the earth as a "plane, on which the meridians are parallel straight "lines; two suppositions which are false and ridicu- "lously absurd. The seaman deluded by these false "notions cannot possibly form a true idea either of "Middle Latitude or of Mercator's sailing; and con- "sequently the principles of his favourite art are for "ever buried under the rubbish of quacks and im- "postors." In England this spurious system has been vigorously attacked by some of the best mathematicians of the present age. See Emerson's Navigation, Emerson's Mathematical Principles of Geography and Navigation, Dr. Hutton's Miscellanea Mathematica, The British Palladium, &c. &c. In America Mr. Baron has, in a small pamphlet, exhibited the genuine principles of Navigation, and completely refuted these errors. This pamphlet is sold in New-York by T. & J. Swords, and in Philadelphia by W. P. Farrand and Co. price 25 cents. Young navigators, not acquainted with the higher branches

of the mathematics might, in my opinion, derive considerable benefit from a careful perusal of that work.

T. M.

VI. QUEST. 78. *Solved by Jacob Winooski.*

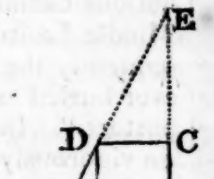
Put r=radius, s=the sine, c=the cosine, and t= the tangent of the required angle; then by Trigonometry, $c : s :: r : \frac{rs}{c} = t$, $= c$ per quest. Whence $rs = c^2 = r^2 - s^2$, or $s^2 + rs = r^2$; from which we find $s = \frac{1}{2} r \times (\sqrt{5} - 1) = .6180340 \times r$, the natural sine of 38° 10′ 22″, the quantity of the required angle.

Corollary. In this angle the secant is equal to the cotangent: for by Trigonometry,

$c : r :: r$: secant,

and $t = c : r :: r$: cotangent.

VII. QUEST. 79. *Solved by Robert Adrain, Reading, Pennsylvania.*



Let ABCD be the trapezoid, having the parallel sides AB and CD, either of which as AB is given, together with its adjacent angles, and the area of the figure ABCD. From D draw DF at right angles to AB. Produce AD and BC to meet in E, and the triangles ABE and DCE are similar and have given angles, and AB is given: whence the side AE, and the area of ABE become known; and the area of ABCD taken from the area of ABE, leaves the area of DCE. Lastly, the area of ABE is to the area of DCE as the square of AE is to the square of DE; whence DE is known, and of course AD and DF are also known.

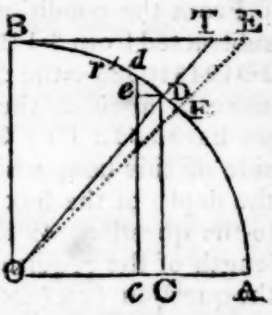
VIII. QUEST. 80. *Solved by the proposer, the Rev. Thomas P. Irving, Newbern, North-Carolina.*

From the conditions of the question, ·1999689615 subtracted from 3·1194 the length of the curve, leaves 2·9194310385=the diagonal of the cubical box. And since in a cube, the diagonal : the side : : $\sqrt{3}$: 1, we have $\sqrt{3}$: 1 : : 2·9194310385 : 1·685534= s the side of this box, which, by the question, is equal to the depth of the box required. Let now, according to the question, $4x$ and $7x$ represent the breadth and length of the required box; then by mensuration and the question $4x \times 7x \times s = 2s^3$, from which $x = s \div \sqrt{14}$ = ·450478 : consequently $4x$ = 1·801912, and $7x$ = 3·153346 inches, the length and breadth of the box required.

IX. QUEST. 81. *Solved by N. Young, near Lexington, Kentucky.*

By a few trials, we readily find by Mercator's Sailing, that the difference of longitude corresponding with latitude 16° 35′ N. (=995′) and the given course, is 987·7′, which is 7·3 too small. Again, by the same means, we find the difference of longitude corresponding with latitude 16° 30′ N. (=990′) and the given course, is 991·2′, which is 1·2 too great. Then by a common rule in Double Position 7·3+1·2 = 8·5 : 1·2 : : 5′ : ·7′, which added to the latitude last supposed, gives 16° 30·7′. Hence a ship from latitude 38° 15′ N. sailing on a S. W. by S. course, will, on arriving in latitude 16° 30·7′ N. have made her difference of longitude equal to her latitude. Lastly, by Plain Sailing, the distance required in the question is found=1568·7′.

X. Prize Quest. 82. *Solved by Peter Delamar, Philadelphia.*



In a circle whose centre is O, let OA and OB be the bounding radii of a quadrant OAB. Let AD = z represent any variable arc, to which OC=x is the cosine, BT the cotangent and ODT the cosecant; draw *de* parallel to and infinitely near DC, and D*e* parallel to OA; then the fluxionary triangle D*de* is similar to the triangles ODC and OBT, D*d*=$\dot{z}$, and D*e* = *c*C=$\dot{x}$. But by the question xz is a maximum; and since, when z increases x decreases, and the contrary, we have $z\dot{x} \circ x\dot{z} = 0$, or $z\dot{x} = x\dot{z}$: whence $z : x :: \dot{z} : \dot{x}$ :: AD : OC :: D*d* : D*e* :: OD : DC. Further, by alternation AD : OD :: OC : DC, and by similar triangles OC : DC :: BT : BO=OD; and consequently the arc AD is equal to the cotangent BT, the relation required in the question.

Corollary 1. Hence, by a similar process, when the rectangle of an arc and its cotangent is a maximum, the relation between the arc and its cosine may be found.

Cor. 2., *By Robert Adrain.* If instead of xz we suppose $x^n z^m$ to be a maximum; we shall find, by proceeding as above, that $m : n ::$ AD : BT : hence it appears, that the present question, is only a particular case of this general corollary.

Cor. 3. *By Robert Adrain.* The measure of the arc AD is easily found. For AD=BT is greater than BD, and therefore greater than $\frac{1}{2}$ AB, or its equal AF. Let OF and BT produced meet in E, and

make the arc $Ar = 57°\cdot3$ = the length of radius = BE; then is Dr = TE. And because a particle of the tangent BE at E is double the corresponding particle of the curve at F, it follows that TE is nearly double DF, and consequently that DF is nearly the third part of Fr. But $Fr = 57°\cdot3 - 45° = 12°\cdot3$; therefore DF is nearly $= 4°\cdot1$, and the arc $AD = 49°\cdot1 = 49°\ 6'$; from which, by a stating with trial and error, we obtain the true measure of the arc $AD = 49°\ 17'\ 36''\cdot6$.

The same solved by Jacob Concklin, jun. Rockland, New-Jersey.

Let z = the length of a variable arc, x = its sine, and its radius = 1; then by Trigonometry its cosine $= \sqrt{1-x^2}$; and by the question $z \times \sqrt{1-x^2}$ is a maximum, whose fluxion reduced gives $\dot{z} = \frac{z\,x\,\dot{x}}{1-xx}$. Now since it is well known that $\dot{z} = \frac{\dot{x}}{\sqrt{1-xx}}$, we have $\frac{\dot{x}}{\sqrt{1-xx}} = \frac{zx\dot{x}}{1-xx}$, from which $z = \frac{\sqrt{1-xx}}{x}$. But by Trigonometry the sine $= x$: radius $= 1$: : the cosine $= \sqrt{1-xx}$: the cotangent $= \frac{\sqrt{1-xx}}{x}$; and hence the length of the arc is equal to that of its cotangent, the required relation.

ACKNOWLEDGEMENTS, &c.

A list of the names and places of residence of the ingenious gentlemen who have given true solutions to questions proposed in No. VIII. The figures in horizontal lines with the names refer to the questions answered by each, as numbered in the work.

Robert Adrain, Reading, Pennsylvania,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10.
G. Baron, New-York.	1, 2, 3, 4, 5, 6, 7, 8, 9, 10.
John Capp, Harrisburg, Pennsylvania,	1, 2, 3, 4, 6, 7.
Jocob Concklin, Rockland, New-Jersey,	1, 2, 3, 4, 5, 6, 7, 10.
John Craggs, near Richmond, Virginia,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10.
John D. Craig, Baltimore,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10.
Peter Delamar, Philadelphia,	10.
John Eberly, Lancaster, Pennsylvania,	7.
John Gaultier, New-Orleans,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10.
Martin Gibson, George-Town, South-Carolina,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10.
John Gummere, near Burlington, N. J.	1, 2, 6, 7.
Rev. T P. Irving, Newbern, North-Carolina,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10.
James M'Ginness, Harrisburg, Pennsylvania,	1, 2, 3, 4, 6, 7.
James M'Cormick, Carlisle, Pennsylvania,	1, 2, 3, 6, 7.
Thomas Maughan, Quebec,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10.
Seth Robson, Savannah, Georgia,	1, 2, 3, 4, 6, 8.
Henry Smith, New-York,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10.
John Smithis, Philadelphia,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10.
Peter Spangler, York-Town, Pennsylvania,	1, 2, 7.
Jacob Winooski, Green Mountains, Vermont,	1, 2, 3. 6, 7.
Thomas Whittaker, York-Town, Pennsylvania,	1, 2, 3, 4, 6, 7, 8.
E. R. White, Danbury, Connecticut,	1, 6,
James Williams, Petersburg, Virginia,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10.
John Wilson, Boston,	1, 2, 3.
Diarius Yankee, Bunker's Hill,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10.
T. Youle, Portland, District of Maine,	1, 2.
Abner Young, New-Haven, Connecticut,	1.
N. Young, near Lexington, Kentucky,	1, 2, 3, 4, 5, 6, 7, 8, 9, 10.

Two silver medals, value six dollars each, have, for the solutions of the *prize question*, been awarded, one to PETER DELAMAR, of Philadelphia, and the other to JACOB CONCKLIN, jun. of Rockland, in New-Jersey. These medals may be had at Mr. Baron's, No. 24 Cedar street, New-York, on the first day of May next.

The present number of this work will not be delivered to subscribers in arrears with the editors. The portrait accompanying this number is intended as a frontispiece, to face the title page of our first volume.

This portrait has been executed by two American artists spontaneously eminent. It was painted by *Mr. Howell* of New-York, a young man born and brought up on Long-Island, who was never instructed in the art of drawing; it was engraved by the celebrated *Dr. Anderson* of New-York, whose talents are, independent of the least instruction whatever, the results of his own genius and industry. *Corollary.* Genius has spontaneously grown on American soil.

It is probable that the first number of our second volume will not appear until the first day of May next; and that, in the mean time each of our subscribers will receive a circular letter on the subject. Mr. Baron, owing to other engagements, cannot possibly be concerned in editing the future numbers of the Mathematical Correspondent.

ARTICLE XXIII. *Continued from page* 186.

¶ 12. The general proof of this principle, or that which extends to all sorts of quantities, both commensurable and incommensurable, is as follows. Since 1 is to m as a to ma, and $1 : m :: b : mb$, and $1 : m :: c : mc$, and consequently $a : ma :: b : mb :: c : mc$, it follows (by El. 5, 12.) that as a to ma, or as 1 to m, so is $a+b+c$ to $ma+mb+mc$; that is, $ma+mb+mc$ is $= \overline{a+b+c} \times m$. *Q. E. D.*

Scholium.

¶ 13. We just now called the demonstration of the first proposition of the fifth book of Euclid's Elements an algebraic one, because the reasoning in it is of the same kind as is used in algebra: and for the same reason the whole fifth book may be called algebraic, and contains perhaps the best demonstrations of the principles of algebra that are any where to be found, the modern writers of algebra having almost universally taken them for granted, without so much as

endeavouring to demonstrate them. Now, that the fifth book of the Elements may be justly said to contain the principles of algebra will be evident, by considering that it is wholly taken up in demonstrating the properties of proportionality; and that as algebra is the art of computation, or of finding and demonstrating the properties of all quantities whatsoever that are capable of proportion, as far as those properties depend upon their magnitudes and proportions, every book that treats of the general properties of proportionality, and the several relations of quantities that arise merely from thence, without any regard to the peculiar properties of any particular kind of quantity, must necessarily contain the principles of algebra: and all that it can possibly want further, to give it the complete appearance of a treatise on algebra, is the notation, or, as Sir Isaac Newton calls it, the language of algebra. And hence we may observe, that all those demonstrations of geometrical propositions that proceed entirely by reasoning upon the magnitudes and proportions of the quantities under consideration, without any regard to their positions or other peculiar properties, are of an algebraic kind, or differ from algebraic demonstrations only in their notation; and indeed the difference between algebra and geometry is much less than it is by many apprehended to be, and consists in most cases merely in the notation; it being but very seldom, and in the most simple and easy propositions that we can, by the mere drawing of a few lines, find or demonstrate the thing in question, without any abstract reasoning upon proportion.

Hence, I think, we may form some kind of judgment concerning a question that has been much debated amongst mathematicians; to wit, whether algebraic or geometrical demonstrations and investigations of geometrical propositions are to be preferred? for when a geometrical proposition can be demonstrated in a simple manner, by drawing a few lines

of an easy construction, and a very little reasoning upon them, and can be derived as it were from the very essence of the figure, so that you keep its generation and fundamental properties in view throughout the whole proof, the foregoing question then amounts to this, whether this proof, or one deduced from the abstract properties of proportionality, and expressed in algebraic characters, is the best? And here I believe it will be generally allowed that the former is to be preferred: but when the proposition admits of no proofs but such as are deduced from the abstract properties of proportionality, by a variety of componendos, dividendos, invertendos, ex æquos, taking continual or intermediate proportionals, &c. the question then is, whether this reasoning had best be expressed in algebra or in common language? and the contest, in this case, is not between two different methods of reasoning, but between two different methods of expressing one and the same reasoning. Now, here it seems reasonable to conclude, that if the steps in the demonstration are but few, 'tis hardly worth while to depart from the common language to express them; but if they are many, I am persuaded it will be found infinitely shorter, and more convenient, to express them in algebraic characters than in words, as they will all lie close together in a narrow compass, uninterrupted by the tedious phrases and circumlocutions which cannot be avoided when we make use of words; nay, there are some things which would be almost, if not wholly, unintelligible, if expressed in words, merely from the immense prolixity they would occasion; as any one will be easily convinced, who shall endeavour to translate Sir ISAAC NEWTON's binomial, or Mr. DEMOIVRE's multinomial theorem, or any others of that kind, out of algebra into common language. However, as algebra is so extremely general a language, that it is frequently incapable of expressing all the conditions and limita-

tions of a problem, care ought always to be taken to supply those deficiencies, by making use of words; otherwise our conclusions will necessarily be too general, and the answers or solutions of the problem more in number than they ought to be.

Sect. III.

Of the Multiplication and Division of Differences.

¶ 14. The two foregoing principles mentioned in ¶ 6. being now established, the following theorem may easily be demonstrated.

Theorem.

The product that arises by multiplying the difference of any two quantities a and b, whereof a is the greater, by a third quantity m is equal to the excess of the product of a into m above the product of b into m; or $\overline{a-b}\times m$ is $= ma-mb$.

For let $c = a-b$; then will $c+b=a$, and consequently $\overline{c+b}\times m=ma$. But $\overline{c+b}\times m$ is $= mc+mb$ by the second of the foregoing principles; therefore $mc+mb$ is $= ma$; whence mc is $=ma-mb$, or $\overline{a-b}\times m$ is $=ma-mb$. *Q. E. D.*

Coroll. 1. The same is true if the multiplicand consists of any number of terms whatsoever, as $a-b+d-e+f$, to wit, that the product that arises by multiplying the whole compound quantity $a-b+d-e+f$ into any number m is equal to the quantity that arises, by first multiplying each of the terms a, b, d, e, f, into m, and then connecting the products together by the signs + and —, or by addition and subtraction, in the same manner as those terms themselves were connected together before multiplication: or, to express the same in algebraic symbols, $\overline{a-b+d-e+f}\times m$ is $= ma-mb+md-me+mf$.

For let $a+d+f=g$, $b+e=h$; then will $a+d+f-b-e$ be $=g-h$, and consequently $\overline{a+d+f-b-e}\times m=\overline{g-h}\times m$. But by this theorem we have $g-h\times m=mg-mh=\overline{a+d+f}\times m-\overline{b+e}\times m=$(by the second of the foregoing principles) $ma+md+mf-mb+me=ma+md+mf-mb-me$; therefore $\overline{a+d+f-b-e}\times m$ is $=ma+md+mf-mb-me$, or $\overline{a-b+d-e+f}\times m$ is $=ma-mb+md-me+mf$. *Q. E. D.*

Coroll. 2. If any quantity as m, is multiplied into a compound multiplicator, as $a-b+d-e+f$, consisting of any number of terms, whereof some are affirmative and some negative, the product thence arising will be the same as would arise by first multiplying each of the terms a, b, d, e, f, of the multiplicator into m, and then connecting the products together by the signs $+$ and $-$, in the same manner as those terms themselves were connected before multiplication: or, to express the same in algebraic symbols, $m\times\overline{a-b+d-e+f}$ is $=ma-mb+md-me+mf$.

For, by the first of the above-mentioned principles, the product of m multiplied into $a-b+d-e+f$ is equal to the product of $a-b+d-e+f$ multiplied into m, that is, by Coroll. 1. to $ma-mb+md-me+mf$.

Coroll. 3. Since multiplying m into each of the terms a, b, d, e, f, produces (by the first of the premised principles) the same products as are produced by multiplying each of those terms into m, it follows that if any quantity, as m, is multiplied into a compound multiplicator, as $a-b+d-e+f$, the product thence arising will be the same as would arise by first multiplying m into each of the terms a, b, d, e, f, of the multiplicator, and then connecting the products together, by the signs $+$ and $-$, in the same manner as those terms themselves were connected before multiplication: or, to express the same in algebraic symbols, $m\times\overline{a-b+d-e+f}$ is $=am-bm+dm-em+fm$.

Coroll. 4. If any compound quantity, as $p-q+r-s+t$, consisting of any number of terms, be multiplied into any compound multiplicator, as $a-b+d-e+f$, the product thence arising will be equal to the quantity which arises by multiplying all the terms of the multiplicand into each term of the multiplicator, and connecting the products together, by the signs + and —, according to this rule, "that the terms of every set of products that arises by multiplying $p-q+r-s+t$ into a term of the multiplicator, marked with the sign +, as the terms a, d, and f, shall have the same signs prefixed to them as were prefixed to the terms of the multiplicand $p-q+r-s+t$; and the terms of every set of products that arises by multiplying $p-q+r-s+t$ into a term of the multiplicator, marked with the sign —, as are the terms b and e, shall be marked with the signs contrary to those of the terms of the multiplicand $p-q+r-s+t$."

For let $p-q+r-s+t=m$. By Corollary 3, $m\times\overline{a-b+d-e+f}$ is $=am-bm+dm-em+fm$; therefore (substituting $p-q+r-s+t$ instead of m) $\overline{p-q+r-s+t}\times\overline{a-b+d-e+f}$ is equal to the sum of the three products of $p-q+r-s+t$ into a, of $p-q+r-s+t$ into d, and of $p-q+r-s+t$ into f, diminished by the sum of the two products of $p-q+r-s+t$ into b, and of $p-q+r-s+t$ into e. These products are (by Coroll. 1.)

1st, $ap-aq+ar-as+at$
2dly, $dp-dq+dr-ds+dt$
3dly, $fp-fq+fr-fs+ft$
4thly, $bp-bq+br-bs+bt$
5thly, $ep-eq+er-es+et$, in all the terms of which the signs are the same as in the terms of the multiplicand $p-q+r-s+t$. The three first of these products are to be added together, which (by ¶ 4.) is performed by connecting their terms together, without altering their signs; which proves the former part of this corollary: and the two latter

products are to be subtracted from the former, and therefore (by ¶ 5.) must be connected with them, with the signs of their terms every where changed; which proves the latter part of the corollary. The product therefore of $p-q+r-s+t$ into $a-b+d-e+f$ is equal to the following set of products;

$$\begin{array}{l} ap-aq+ar-as+at \\ +dp-dq+dr-ds+dt \\ +fp-fq+fr-fs+ft \\ -bp+bq-br+bs-bt \\ -ep+eq-er+es-et \end{array}$$

¶ 15. 'Tis evident the preceding rule coincides with the common one mentioned in ¶ 3; to wit, that $-\times+$ gives $-$, $+\times-$ gives $-$, $+\times+$ gives $+$, and $-\times-$ gives $+$; or, that as often as the signs of the factors are alike the product must be marked with the sign $+$, and as often as they are unlike, the product must be marked with the sign $-$. For in multiplying $p-q+r-s+t$ by a, d, and f, which are marked with the sign $+$, the terms of the product have respectively the same terms of the multiplicand $p-q+r-s+t$; therefore, as often as the terms of the multiplicand are marked with the sign $+$, or the signs of the factors are alike, the products are marked with the sign $+$; and as often as the terms of the multiplicand are marked with the sign $-$, or the signs of the factors are unlike, the products are marked with the sign $-$. And in multiplying $p-q+r-s+t$ by b and e, which are marked with the sign $-$, the terms of the product are marked with signs respectively contrary to those of the terms of the multiplicand $p-q+r-s+t$; therefore, as often as the terms of the multiplicand are marked with the sign $+$, or the signs of the factors are unlike, the products are marked with the sign $-$; and as often as the terms of the multiplicand are marked with the sign $-$, or the signs of the factors are alike, the products are marked with the sign $+$.

¶ 16. Some of these cases of multiplication are ex-

pressly contained in the fifth book of the Elements, as will be evident to any one that shall examine it; which that the reader may do it with more ease, I shall here set down most of the propositions contained in it in algebraic characters, as follows. Let a and b represent any two quantities whatsoever of the same kind, whether commensurable or incommensurable, whereof a is the greater; p a whole number, or multiple of the quantity called unit; n a whole number, in the 2d, 3d, 4th, and 6th propositions, and in the 22d, 23d, and 24th, any quantity whatsoever, whether commensurable or incommensurable, to a unit; m a whole number, in the 1st, 2d, 3d, 4th, 5th, 6th, and 15th propositions, and in the rest, any quantity whatsoever, whether commensurable or incommensurable, to a unit;

And the first proposition will (as has been already observed in ¶ 11.) be this, $ma+mb=m\times\overline{a+b}$;

The 2d will be this, $ma+na : a :: mb+nb : b$;

The 3d this, $mna : a :: mnb : b$;

The 4th this, $pma : na :: pmb : nb$;

The 5th this, $ma-mb=m\times\overline{a-b}$;

The 6th this, $ma-na : a :: mb-nb : b$;

The 12th this, $ma : a :: ma+mb+mc : a+b+c$;

The 15th this, $a : b :: ma : mb$;

The 16th this, $ma : mb :: a : b$;

The 17th this, $ma-a : a :: mb-b : b$;

The 18th this, $ma+a : a :: mb+b : b$;

The 19th this, $ma : a :: ma-mb : a-b$;

The corollary to the 19th this, $ma : ma-a :: mb : mb-b$;

The 22d this, $ma : na :: mb : nb$;

The 23d this, $ma : na :: \frac{mb}{n} : b$;

And the 24th this, $ma+na : a :: mb+nb : b$.

The demonstrations of these propositions might here be expressed in the same characters; but this, as it is not absolutely necessary, after what has been

already demonstrated in the foregoing theorem and corollaries, and may very easily be done by any one that pleases to try it, I shall leave to the industry and curiosity of the reader.

¶ 17. As it has been shown in ¶ 9. that the product of two quantities is exactly the same, whichsoever of them is made the multiplicator, I shall, in the following articles, make no kind of distinction in the notation of such products, according as one or other of the factors is considered as the multiplicator, but shall denote the product of p and q indifferently by pq or qp, whether p be multiplied by q, or q by p, as I imagine such a distinction cannot be of any use, although, for the sake of perhaps a needless accuracy in our first entrance on the subject of these articles, it has been observed in the foregoing theorem and corollaries.

¶ 18. It will be useful to add the following instances of multiplication.

$$\begin{array}{r} a+b \\ a+b \\ \hline aa+ab \\ \quad +ab+bb \\ \hline aa+2ab+bb \\ \quad\ \ a+b \\ \hline a^3+2a^2b+ab^2 \\ \quad +a^2b+2ab^2+b^3 \\ \hline a^3+3a^2b+3ab^2+b^3 \end{array} \qquad \begin{array}{r} a-b \\ a-b \\ \hline aa-ab \\ \quad -ab+bb \\ \hline aa-2ab+bb \\ \quad\ \ a-b \\ \hline a^3-2a^2b+ab^2 \\ \quad -a^2b+2ab^2-b^3 \\ \hline a^3-3a^2b+3ab^2-b^3 \end{array}$$

$$\begin{array}{r} a+b \\ a-b \\ \hline aa+ab \\ \quad -ab-bb \\ \hline aa\ \ .\ \ -bb \end{array} \qquad \begin{array}{r} aa-ab+bb \\ \quad\ \ a+b \\ \hline a^3-a^2b+ab^2 \\ \quad +a^2b-ab^2+b^3 \\ \hline a^3\ \ .\ \ .\ \ +b^3 \end{array}$$

$$\begin{array}{r} aa+ab+bb \\ \underline{\qquad\qquad a-b} \\ a^3+a^2b+ab^2 \\ \quad -a^2b-ab^2-b^3 \\ \hline a^3 \quad . \quad . \quad -b^3 \end{array}$$

From these operations it appears,

1st. That the square of the sum of any two quantities is equal to the sum of the squares of those quantities increased by twice their product; which answers to the 4th proposition of the 2d book of the Elements.

2dly. That the cube of their sum is equal to the sum of their cubes increased by the quantity that arises by multiplying their sum into three times their product. For $3a^2b+3ab^2$ is $=\overline{a+b}\times 3ab$.

3dly. That the square of their difference is equal to the sum of their squares diminished by twice their product. This conclusion corresponds to the property of geometrical squares and rectangles, demonstrated in the 7th proposition of the 2d book of Euclid: for it is there proved, that if any line denoted by the letter a be divided into two parts b and $a-b$, the square of the whole line a, together with the square of the part b, is equal to twice the rectangle contained under the whole line a and the part b, together with the square of the other part $a-b$; that is, $aa+bb=2ab+\overline{a-b}^2$; consequently $\overline{a-b}^2$ is $=aa-2ab+bb$.

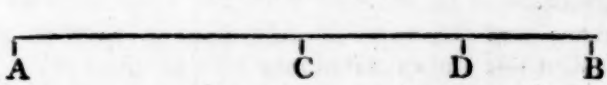
Therefore, 4thly. The sum of the squares of any two quantities is greater than twice their product.

5thly. The cube of the difference of any two quantities is equal to the difference of their cubes diminished by the quantity which arises by multiplying their difference into three times their product. For $\overline{a-b}^3$ $(=a^3-3a^2b+3ab^2-b^3=a^3-b^3-\overline{3a^2b-3ab^2})$ is $=a^3-b^3-3ab\times\overline{a-b}$.

Therefore, 6thly. The difference of their cubes must always be greater than that quantity, or $a^3 - b^3$ must always be greater than $3ab \times \overline{a-b}$.

7thly. The product of the sum, and difference of any two quantities, is equal to the difference of their squares. This answers to the 5th proposition of the 2d book of EUCLID; for it is there proved, that if any line, as AB, is divided into two unequal parts AD, DB, and also into two equal parts AC, CB,

FIGURE IV.



the rectangle under the two unequal parts AD, DB, together with the square of the line CD that lies between the sections, is equal to the square of the half, or of CB. Now, 'tis evident that CD is less than CB, that AD (the greater of the unequal parts) is equal to the sum of AC and CD, or CB and CD, and DB (the lesser of the unequal parts) to their difference; therefore $\overline{CB+CD} \times \overline{CB-CD} + CD^2$ is $= CB^2$; and consequently $\overline{CB+CD} \times \overline{CB-CD} = CB^2 - CD^2$. It may be useful to observe, that this proposition of EUCLID proves that if a given line AB be supposed to be described by the motion of the point D from B towards A, the rectangle under the segments AD, DB, will increase continually, till the point D coincides with the point C; for so long the square of CD decreases: afterwards this rectangle will decrease continually while D is passing from C to A: for, during that time, the square of CD increases again till it equals the square of CA: and the greatest magnitude this rectangle ever attains is the square of CB, or of half the given line AB.

8thly. The product that arises by multiplying together the sum of two quantities and the sum of their

squares diminished by their product, is equal to the sum of their cubes.

And, 9thly. The product that arises by multiplying together the difference of two quantities, and the sum of their squares increased by their product, is equal to the difference of their cubes.

These, and other like observations, are very frequently of use in algebraic computations.

¶ 19. In division the rule for determining the signs of the terms of the quotient is as follows: "Whenever the dividend is marked with the sign +, the quotient must be marked with the same sign as the divisor; and whenever the dividend is marked with the sign —, the quotient must be marked with the sign that is contrary to that of the divisor." For the quotient multiplied into the divisor must produce the dividend.

ARTICLE XXVI.

A View of the Diophantine Algebra. By Robert Adrain.

¶ 1. We may define the general object of this inquiry to be *the method of finding such rational values of one or more indeterminate quantities, that any proposed functions of those quantities may become rational squares, or cubes, &c.* It will save frequent repetition to remark here, that all the numbers given or required in this subject are understood to be rational.

¶ 2. *Let us begin with the formula* $ax+b$; *a, and b, being any given numbers.*

As an example of this formula, suppose it be required to assign such a value for x that $2x+3$ may be a square number. Here, if we substitute 1 for x, $2x+3$ becomes $2+3$ or 5, which is not a square, and therefore 1 is none of the required values of x; if $x=2$, we have $2x+3=7$, which is not a square; but if $x=3$, we shall have $2x+3=6+3=9$ which is evidently a square number, and therefore 3 is one of

the required values of x. It is easy to find as many of these values of x as we please, merely by equating the proposed formula successively to any number of known squares. Thus, suppose $2x+3=25$, whence $2x=22$, and therefore $x=11$, which is one of the values sought. But we may have a general expression for all the values of x by assuming $2x+3=a^2$; from this we have $2x=a^2-3$, and $x=\frac{a^2-3}{2}$: we may now assume a of any value at pleasure, and we shall obtain successively all the values of x that can render $2x+3$ a rational square.

On the very same principle we may find in a general manner all the possible values of x that will render $ax+b$ a rational square; we have only to suppose $ax+b=c^2$, from which we obtain $x=\frac{c^2-b}{a}$.

¶ 3. *Let* $\frac{ax+b}{cx+d}$ *be made a square, a, b, c, and d, being any given numbers.*

This formula is equally easy with the preceding; we have only to suppose $r=$the root of the square, which furnishes us with the equation $\frac{ax+b}{cx+d}=r^2$, from this we obtain $x=\frac{b-dr^2}{cr^2-a}$, which is a general expression for all the possible values of x that can answer the question, by assuming for r successively all possible numbers.

Any particular case of the formula $\frac{ax+b}{cx+d}$ is made a square in the same manner; for example, let us find the values of x which will make $\frac{1+x}{1-x}$ a square. Assume $\frac{1+x}{1-x}=r^2$, and from this we obtain $x=\frac{r^2-1}{r^2+1}$, in which r may be taken at pleasure: if we suppose

$r=2$, we shall have $x=\frac{4-1}{4+1}=\frac{3}{5}$, and $\frac{1+x}{1-x}=\frac{1+\frac{3}{5}}{1-\frac{3}{5}}=4$, which is a square,

¶ 4. *Let* $(ax+b)\times(cx+d)$ *be made a square; a, b, c, and d, being as usual given numbers.*

This formula is reducible to that in the preceding paragraph by remarking that any square multiplied or divided by a square will always produce a square: if therefore $(ax+b)\times(cx+d)$ be divided by $(cx+d)^2$, the quotient $\frac{ax+b}{cx+d}$ will always be a square when the proposed formula is a square, and vice versa. Therefore we have only to assume $\frac{ax+b}{cx+d}=r^2$, and consequently $x=\frac{b-dr^2}{cr^2-a}$ as in the preceding paragraph.

But the formula $(ax+b)\times(cx+d)$ may be made a square without reducing it to $\frac{ax+b}{cx+d}$: this is effected by supposing the root of the square to be $(cx+d)\times r$, which gives the equation $(ax+b)\times(cx+d)=(cx+d)^2\times r^2$; dividing by $cx+d$ we have $ax+b=(cx+d)\times r^2$, and consequently $x=\frac{b-dr^2}{cr^2-a}$ as before.

As an example of this, let $1-x^2$ or $(1+x)\times(1-x)$ be made a square. Assume $(1+x)\times(1-x)=(1-x)^2\times r^2$, dividing by $1-x$ we have $1+x=r^2-r^2x$, and therefore $x=\frac{r^2-1}{r^2+1}$, and r may be taken at pleasure. Suppose $r=2$, this gives $x=\frac{3}{5}$, and $1-\frac{9}{25}=\frac{16}{25}=$ a square; or suppose $r=3$, which gives $x=\frac{4}{5}$. In the same manner we may assign the values of x which will make x^2-a^2 a rational square, a being any given number.

¶ 5. *Let the formula of the second degree* ax^2+bx+c *be made a rational square.*

We commence this solution with remarking that

the proposed formula is not susceptible of a universal solution, in the same manner as those we have already considered: we may even assign such values of a, b, and c that ax^2+bx+c can never become a square by assuming for x all possible numbers; a universal solution therefore in terms of a, b, and c, with given or assumed quantities cannot be expected. And even when a, b and c are of such values that the formula may be a square, we are frequently at a loss to assign the values of x without having recourse to methods more or less tentative. There are however saveral general cases in which we obtain the values of x with great facility: of these we shall endeavour to give some account.

If the absolute number c be wanting, the formula becomes ax^2+bx; let rx be the root of the square, and we have $ax^2+bx=r^2x^2$, which divided by x gives $ax+b=r^2x$, and therefore $x=\frac{b}{r^2-a}$. For example, let us find x such that $x-x^2$ may be a square. Assume $x-x^2=r^2x^2$, from which we obtain $x=\frac{1}{r^2+1}$, and r may be taken at pleasure: if $r=1$, $x=\frac{1}{2}$ and $x-x^2$ becomes $\frac{1}{2}-\frac{1}{4}=\frac{1}{4}$ which is manifestly a square.

If the equation $ax^2+bx+c=0$ has rational roots, the values of x are readily found that will render the proposed formula a square; for in that case the formula ax^2+bx+c will always be the product of two rational factors as $(mx+n)\times(px+q)$, which we have only to equate to $(px+q)^2\times r^2$; from this we obtain by division $mx+n=r^2px+r^2q$, and therefore $x=\frac{r^2q-n}{m-r^2p}$. Example, suppose x^2+3x+2 is to be a square. By resolving the equation $x^2+3x+2=0$, we find its two roots are -1 and -2, and thus we discover that $x^2+3x+2=(x+1)\times(x+2)$, which equated to $(x+2)^2\times r^2$ gives $(x+1)=(x+2)r^2$, whence $x=\frac{2r^2-1}{1-r^2}$, and r may be taken at pleasure.

Assume $r=\frac{4}{5}$, and we find $x=\frac{7}{0}$, which value substituted for x gives $x^2+3x+2=\frac{49}{81}+\frac{21}{0}+2=\frac{49}{81}+\frac{189}{81}+\frac{162}{81}=\frac{400}{81}=(\frac{20}{9})^2$. In the same manner we discover the values of x that can make $6x^2-13x+6$ a complete square.

¶ 6. If the coefficient of x^2 be a square, we may write the formula thus, a^2x^2+bx+c, and we are able to find all the values of x that can render this expression a square. For this purpose we are to assume $ax+r$ for the root of the square, which gives this equation $a^2x^2+bx+c=a^2x^2+2arx+r^2$, from which we have $bx+c=2arx+r^2$, and thence $x=\frac{c-r^2}{2ar-b}$; and r may be assumed at pleasure either positively or negatively.

To explain this let us apply the method to the formula $4x^2+5x+6$. Assume $2x+r$ for the root which gives us the equation $4x^2+5x+6=4x^2+4rx+r^2$, whence $5x+6=4rx+r^2$ and consequently $x=\frac{6-r^2}{4r-5}$. Now take $r=-3$, whence $x=\frac{3}{17}$ which on trial will no doubt be found to succeed.

On this principle we may find the values of x that will render x^2+2 a square. We have only to suppose $x^2+2=(x+r)^2=x^2+2rx+r^2$, whence $2=2rx+r^2$, and $x=\frac{2-r^2}{2r}$; if $r=1$, we have $x=\frac{1}{2}$.

¶ 7. When the absolute number is a square, the formula may be written thus, ax^2+bx+c^2. To make this a square, suppose $rx+c$ to be the root, and we shall have $ax^2+bx+c^2=r^2x^2+2crx+c^2$; and c^2 being taken from both sides, the resulting equation $ax^2+bx=r^2x^2+2crx$ is divisible by x, from which there is found $ax+b=r^2x+2cr$, and consequently $x=\frac{2cr-b}{a-r^2}$.

Let $5x^2-x+1$ be an example. Here we are to assume $5x^2-x+1=(rx+1)^2=r^2x^2+2rx+1$, whence

$5x^2-x=r^2x^2+2rx$, which divided by x gives $5x-1=r^2x+2r$, and $x=\frac{2r+1}{5-r^2}$, and r may be assumed of any value we please; if $r=2$, we have $x=\frac{5}{1}$, which value we may substitute for x in $5x^2-x+1$, and try if this expression becomes a perfect square.

Again, let $2x^2+1$ be made a square; assume $2x^2+1=\overline{rx+1}^2=r^2x^2+2rx+1$, and by reduction we obtain $x=\frac{2r}{2-r^2}$. If we take $r=1$, we find $x=2$, which makes $2x^2+1=9$ which is a complete square.

¶ 8. Sometimes we find the general expression ax^2+bx+c under one of the following forms; I. $a^2+(bx+c)\times(dx+e)$. II. $a^2x^2+(bx+c)\times(dx+e)$. III. $(ax+b)^2+(cx+d)\times(ex+f)$. The first of these formulas is made a square by supposing its root $a+r(dx+e)$; the second by supposing its root $ax+r(dx+e)$; and the third by supposing its root $ax+b+r(ex+f)$.

As an example of the first of these, let it be required to find such a value for x that $2-x^2$ may be a square. This expression is evidently equal to $1+1-x^2=1+(1-x)\times(1+x)$; and supposing its root to be $1+r(1+x)$, we have $1+(1-x)\times(1+x)=1+2r(1+x)+r^2(1+x)^2$: each side diminished by unity, and the remainders divided by $1+x$, gives $1-x=2r+r^2+r^2x$, whence $x=\frac{1-2r-r^2}{1+r^2}$. If $r=-1$, we have $x=1$, and $2-x^2=1=$a square; if $r=-2$, we have $x=\frac{1}{5}$, and $2-x^2=2-\frac{1}{25}=\frac{49}{25}$.

To exemplify the second form, let $2x^2-4$ be made a square. Because $2x^2-4=x^2+x^2-4=x^2+(x+2)\times(x-2)$, we are to suppose the root $x+r(x-2)$; and therefore $x^2+(x+2)\times(x-2)=x^2+2rx(x-2)+r^2(x-2)^2$; this equation diminished on each side by x^2; and then divided by $x-2$, gives $x+2=2rx+r^2(x-2)$, from which we obtain $x=\frac{2r^2+2}{r^2+2r-1}$. Now

assume $r=1$, and we shall find $x=2$, which substituted for x gives $2x^2-4=4=2^2$. If $r=\frac{1}{2}$, the value of x will be found $=10$, and $2x^2-4=200-4=196=(14)^2$.

Now let us apply the principle of the third form to $2x^2+17$. It is manifest that $2x^2+17=x^2+9+x^2+8=x^2+6x+9+x^2-6x+8=(x+3)^2+(x-4)\times(x-2)$. and therefore we are to assume the root $=x+3+r(x-2)$ which squared and equated to $(x+3)^2+(x-4)\times(x-2)$, we shall find after reduction that $x=\frac{2r^2-6r-4}{r^2+2r-1}$, and r may be taken at pleasure. Suppose $r=-\frac{1}{2}$, this supposition gives $x=2$, which put for x gives $2x^2+17=8+17=25=5^2$. Again, let $r=1$, whence $x=-4$; but as only the square of x enters $2x^2+17$, we may equally suppose $x=+4$, and $2x^2+17=32+17=49=7^2$. If $r=\frac{1}{2}$ we find $x=-26$, we may therefore take $x=26$, and this will not fail to make $2x^2+17$ a complete square.

¶ 9. When we cannot reduce a quadratic formula to one of the cases mentioned in the preceding sections, we must try successively several numbers for x until we find one value of it which succeeds in rendering the proposed formula a square; and from this value of x we may easily find, by a direct process, all the other values of it which can fulfil the required condition.

For example, let $3x^2+4x+5$ be made a square. This expression is indeed reducible to the 3d. form of ¶ 9th; but if we should be supposed not to have discovered that, we should begin by supposing x successively equal to 1, 2, 3, &c. until we found a suitable value of x: if $x=1$, then $3x^2+4x+5=12$, which is not a square; if $x=2$, we have $3x^2+4x+5=12+8+5=25=$a square, and therefore 2 is one of the values of x. To obtain the other values of x, put $x=2+y$, this substituted for x gives $3x^2+4x+5=25+16y+3y^2$, of which the absolute term 25 is a square, and therefore we may assume the root $=5+ry$; this

squared and equated to $25+16y+3y^2$, will produce $y=\frac{16-10r}{r^2-3}$. If $r=-2$, we find $y=36$, and $x=38$, a number which substituted for x will doubtless render $3x^2+4x+5$ a complete square.

In general let m be such a value of x as will make $ax^2+bx+c=n^2=$a square, and in order to find all the other values assume $x=m+y$; this substituted for x gives $ax^2+bx+c=am^2+bm+c+(2am+b)y+ay^2$, and by supposition $am^2+bm+c=n^2$, therefore $ax^2+bx+c=n^2+(2am+b)y+ay^2$: this expression having now the first term a square has no farther difficulty, we have only to assume the root$=n+ry$, which squared and equated to $n^2+(2am+b)y+qy^2$, gives a general expression for all the values of y; and this expression increased by m gives the general value of x.

¶ 10. We have already observed that the formula ax^2+bx+c cannot possibly become a square when a, b, and c, have certain values, whatever numbers we assume as values of x: there are in fact many such cases, and we cannot persuade ourselves to omit altogether the method of reasoning which teaches us this very singular property of numbers; but we must content ourselves with examining only one or two of the simplest cases.

The fundamental principles on which this inquiry is founded, are the obvious and well known properties of integer numbers with respect to their divisors and remainders. If any integer N be divided by 2, there will either be no remainder, or the remainder will be 1; if therefore we express the integral quotient by n, the number N will be denoted by $2n$ in the former case, and by $2n+1$ in the latter: it is evident then that every integer must belong to one of the forms $2n$, or $2n+1$.

If we consider N with respect to its divisor 3, we shall easily see that there will either be no remainder, or the remainder will be 1 or 2; so that every integer

must belong to one of the forms $3n$, $3n+1$, or $3n+2$. It is no less evident that every integer must belong to one of the four following forms, $4n$, $4n+1$, $4n+2$, $4n+3$.

Let us now seek the form to which N^2 belongs when the integer N itself is in any one of the forms $3n$, $3n+1$, $3n+2$. If N be expressed by $3n$, N^2 will be expressed by $9n^2$, which is not only divisible by 3, but even by the square of 3; in this case therefore N^2 belongs to the form $3n$. If N be expressed by $3n+1$, its square N^2 will be denoted by $9n^2+6n+1$, which divided by 3 leaves 1, and therefore in this case N^2 is of the form $3n+1$. If N be denoted by $3n+2$, its square will be denoted by $9n^2+12n+4$, which divided by 3 leaves 1 as in the preceding case, and therefore N^2 is also in this case of the form $3n+1$. Hence it is evident that every square integer is either divisible by 3, or the remainder is 1, and never 2, so that there is not any square integer whatever which divided by 3 leaves a remainder of 2; or, which is the same thing, the integer form $3n+2$ never can be a square.

If we examine the squares of numbers with respect to the divisor 4, we shall find that all squares belong to the forms $4n$ and $4n+1$, and therefore there can be no integer squares of the form $4n+2$, or $4n+3$: and when we consider squares with respect to the divisor 5, we find that they are all comprehended in the forms $5n$, $5n+1$, and $5n+4$; and therefore, no integer of the form $5n+2$, or $5n+3$ can ever be a square number.

¶ 11. Let us now demonstrate that *the formula* $3x^2+2$ *cannot possibly be a square, whatever values, integral or fractional, we assign to the number* x.

When x is an integer the thing is manifest, for in this case it is plain that $3x^2+2$ is an integer such that being divided by 3 the remainder is 2; and we have just demonstrated in the last paragraph that no such integer can possibly be a square.

But even by taking fractional values for x, the formula $3x^2+2$ cannot become a square. For if possible let $x=\frac{y}{z}$; y and z being integers, and the fraction $\frac{y}{z}$ in its lowest terms; and $3x^2+2=\frac{3y^2}{z^2}+2=\frac{3y^2+2z^2}{z^2}$ is to be a square; but the denominator z^2 is already a square; we have only to show therefore that the numerator $3y^2+2z^2$ cannot possibly be a square.

For if z be divisible by 3, y is not divisible by 3, because the fraction $\frac{y}{z}$ was supposed to be in its lowest terms; whence it evidently follows that when z is divisible by 3, the formula $3y^2+2z^2$ is divisible by 3, but not by 9, because $2z^2$ is divisible by 9 and $3y^2$ is not: but every square number which is divisible by 3 is also divisible by 9, and therefore when z is divisible by 3 the formula $3y^2+2z^2$ cannot possibly be a square.

If z is not divisible by 3, it must be expressed either by $3n+1$, or $3n+2$. If $z=3n+1$, we have $3y^2+2z^2=3y^2+18n^2+12n+2$, which being divided by 3, leaves the remainder 2; but we have demonstrated that no such number can possibly become a square.

If $z=3n+2$, we have $3y^2+2z^2=3y^2+18n^2+24n+8$, which divided by 3 also leaves the remainder 2; neither in this case therefore can $3y^2+2z^2$ possibly be a square. We see therefore that the proposed integral formula $3y^2+2z^2$ is always deficient in the essential property of a square; for when it is divisible by 3 it is not divisible by 9, and when it is not divisible by 3 it leaves the remainder 2, neither of which properties can possibly belong to any square number. It is fully demonstrated therefore that the formula $3x^2+2$ cannot possibly be a square, whatever integral or fractional value we assign to x.

Every step of the preceding demonstration is equally applicable to the formula $3y^2+(3n+2)z^2$, in which n is any integer taken either positively or negatively. If we take for n successively the numbers 1, 2, 3, 4, &c. we find that none of the formulas $3y^2+5z^2$, $3y^2+8z^2$, $3y^2+11z^2$, &c. can ever be a square; or, which is the same in effect, dividing by z^2, we learn that $3x^2+5$, $3x^2+8$, $3x^2+11$, $3x^2+14$, &c. can never be squares. If n be interpreted negatively, we find that the formulas $3x^2-1$, $3x^2-4$, $3x^2-7$, $3x^2-10$, &c. can never be squares.

¶ 12. By considering numbers with respect to the divisor 5, we prove in a similar manner that neither of the expressions $5y^2+2z^2$, or $5y^2+3z^2$, can be a square; or more generally the formulas $5y^2+(5n+2)z^2$, and $5y^2+(5n+3)z^2$ can never become squares, n being understood either positively or negatively; and therefore the expressions $5x^2+5n+2$, and $5x^2+5n+3$ can never be squares. By assuming n successively equal to the series of natural numbers 1, 2, 3, &c. taken positively and negatively, we shall find two series of expressions proceeding both ways ad libitum, none of which can ever be a square.

In addition to this paragraph we may observe that the formula ax^2+bx+c is easily transformed into another having only two terms: this is done by assuming $ax^2+bx+c=\frac{y^2}{4}$ which is a square, from this we obtain $2ax+b=\sqrt{ay^2+b^2-4ac}=\sqrt{ay^2+d}$, putting $d=b^2-4ac$; if therefore we would discover the soluble or insoluble cases of ax^2+bx+c, we have only to examine the formula $ay^2\mp d$. We would willingly show in this place a general method of making $ay^2\mp d$ a square in all possible cases; but the length of the discussion would be very unsuitable to the limits and plan of the present sketch, in which we wish to comprehend as many as possible of the general rules belonging to the subject.

¶ 18. *Let* $a+bv+cv^2+dv^3$ *be made a complete square.* The only case of this formula which we can resolve in a direct manner, is that in which the first term is a square: in this case we may write the formula thus, $a^2+bv+cv^2+dv^3$. Assume $a+rv$ for the root, which squared and equated with the proposed expression, furnishes us with the equation $a^2+bv+cv^2+dv^3=a^2+2arv+r^2v^2$: taking a^2 from each side we have $bv+cv^2+dv^3=2arv+r^2v^2$, and dividing by v there results $b+cv+dv^2=2ar+r^2v$; but still v^2 remains in this equation: in order therefore to determine v by a simple equation, let us assume $2ar=b$, or $r=\frac{b}{2a}$; hence $cv+dv^2=r^2v$, and dividing by v, we have $c+dv=r^2$, and $v=\frac{r^2-c}{d}=\frac{b^2-4a^2c}{4a^2d}$.

We might also assume three terms in the root, as $a+rv+sv^2$; this squared and equated with the proposed formula gives $a^2+bv+cv^2+dv^3=a^2+2arv+(r^2+2as)v^2+2rsv^3+s^2v^4$: now let us assume $2ar=b$, and $r^2+2as=c$, and the equation by subtracting equals, becomes $dv^3=2rsv^3+s^2v^4$, from which after dividing by v^3, we have $v=\frac{d-2rs}{s^2}$.

Example. Let $1+2v+3v^2+4v^3$ be made a rational square. Let us first assume the root of two terms, which must in this case be $1+v$, whence $1+2v+3v^2+4v^3=1+2v+v^2$; this produces the equation $3v^2+4v^3=v^2$, and dividing by v^2, we have $3+4v=1$, which gives $v=-\frac{1}{2}$, and this negative value of v will render $1+2v+3v^2+4v^3$ a complete square.

But when we admit three terms in the root, it must in the present example be $1+v+v^2$, from which we obtain $1+2v+3v^2+4v^3=1+2v+3v^2+2v^3+v^4$; consequently $4v^3=2v^3+v^4$, whence $4=2+v$, and therefore $v=2$, which value may be substituted for v in the formula $1+2v+3v^2+4v^3$, and we shall see if the result be a rational square.

Patrick Nicholson

If we wish to obtain new values of v, we must assume $v=\frac{1}{2}+y$, or $v=2+y$, each of which substituted for v in the proposed formula will produce expressions, from which we may obtain values of y, and consequently new values of v. If we assume $v=-\frac{1}{2}+y$, we shall have $1+2v+3v^2+4v^3=\frac{1}{4}+2y-3y^2+4y^3$, which must be a square: therefore assuming $\frac{1}{4}+2v$ for its root, we have $\frac{1}{4}+2y-3y^2+4y^3=\frac{1}{4}+2y+4y^2$; whence $4y^3-3y^2=4y^2$, and $4y-3=4$, and $y=\frac{7}{4}$, and $v=-\frac{1}{2}+y=\frac{5}{4}$; and therefore $\frac{5}{4}$ substituted for v will render $1+2v+3v^2+4v^3$ a complete square.

But if we assume $\frac{1}{2}+2y+ry^2$ for the square root of $\frac{1}{4}+2y-3y^2+4y^3$, we shall have $\frac{1}{4}+2y-3y^2+4y^3=\frac{1}{4}+2y+(4+r)y^2+4ry^3+r^2y^4$; now assume $4+r=-3$, or $r=-7$, and we have $4y^3=49y^4-28y^3$, therefore $4=49y-28$, and $y=\frac{32}{49}$, and therefore $-\frac{1}{2}+\frac{32}{49}=\frac{15}{98}=v$; if therefore we substitute $\frac{15}{98}$ for v in the formula $1+2v+3v^2+4v^3$, it will doubtless become a perfect square.

I leave it to the learner to discover new values of v by the substitution of $2+y$; the process being altogether similar to what has just been exhibited. It is obvious that by substituting this last found value of v with w, as those before with y, we may again find other new values of v, and so on at pleasure.

When the first term of a cubic formula is not a square, we must begin by seeking among the small numbers for a value of v which will fulfil the required condition; and if we succeed in discovering such a value, we may in general find as many others as we please, by substituting for v the value thus found with a new unknown quantity in the same manner as in the preceding example.

Sometimes however the discovery of one value of v is of no avail in assisting us to find others; of this we have a remarkable example in the formula $1+v^3$, which becomes a square, when $v=2$; but notwithstanding this we are not able to find any other value

of v which will render the proposed formula a perfect square. It frequently happens also that all our labour is lost in trying to discover one value of v that will render a cubic formula a square. We cannot discover any value whatever of v, that will render $v+v^3$, or $v-v^3$ a complete square: but this is no more than what ought to happen, for in fact there are innumerable cubic formulas which cannot possibly become squares; but the arguments by which we may convince ourselves of this impossibility are too tedious to have a place in the present paper.

¶ 14. *Let us make* $a+bv+cv^2+dv^3+ev^4$ *a rational square.*

We commence with the case in which the first term is a square; in this case we have to make a complete square of the formula $a^2+bv+cv^2+dv^3+ev^4$.

Suppose $a+rv+sv^2$ to be its root; this squared is $a^2+2arv+(r^2+2as)v^2+2rsv^3+s^2v^4$ which is to be equal to $a^2+bv+cv^2+dv^3+ev^4$: assume therefore $2ar=b$, and $r^2+2as=c$, which equations determine the values of r and s; we must therefore have $2rsv^3+s^2v^4=dv^3+ev^4$, which divided by v^3 gives $2rs+s^2v=d+ev$, and consequently $v=\frac{d-2rs}{s^2-e}$.

Let us now consider the case in which the coefficient of v^4 is a square, the formula being $a+bv+cv^2+dv^3+e^2v^4$. This case is easily reducible to the preceding by substituting $\frac{1}{y}$ for v, and multiplying the resulting equation by y^4, which produces $e^2+dy+cy^2+by^3+ay^4$; and it is evident this formula is made a square in the same manner as in the preceding case. But we may find a value of v that will satisfy the required condition without this transformation; we have only to denote the root of the square by $s+rv+ev^2$, and assume proper values of r and s. This root squared is $s^2+2rsv+(r^2+2es)v^2+2erv^3+e^2v^4$, which must be equal to $a+bv+cv^2+dv^3+e^2v^4$:

now let us assume $2er=d$, and $r^2+2es=c$, which equations determine the values of r and s; and we shall have $s^2+2rsv=a+bv$, from which is found $v=\frac{a-s^2}{2rs-b}$.

If both the first and last terms be already squares as in the formula $a^2+bv+cv^2+dv^3+e^2v^4$, we may not only obtain two values of v by applying the methods given in the preceding cases; but we may also find four other values of v by new artifices made use of in contriving the root.

In the first place we may assume $a+\frac{bv}{2a}+ev^2$ for the root, which squared and equated with the given expression, presents us with this equation, $a^2+bv+cv^2+dv^3+e^2v^4=a^2+bv+\left(\frac{\frac{1}{4}b^2}{a^2}+2ae\right)v^2+\frac{be}{a}v^3+e^2v^4$, and subducting equals, we have $cv^2+dv^3=\left(\frac{\frac{1}{4}b^2}{a^2}+2ae\right)v^2+\frac{be}{a}v^3$, and dividing by v^2, we have $c+dv=\frac{\frac{1}{4}b^2}{a^2}+2ae+\frac{be}{a}v$, from which is obtained $v=\frac{\frac{1}{4}b^2+2a^3e-a^2c}{a^2d-abe}$.

Secondly, another suitable root is discovered by remarking that the squares of $a+\frac{bv}{2a}+ev^2$ and of $a+\frac{bv}{2a}-ev^2$ have equally the last term e^2v^4 positive; we may therefore assume $a+\frac{bv}{2a}-ev^2$ for the root, which squared and equated with the proposed expression, furnishes us after subtraction with the equation $cv^2+dv^3=\frac{\frac{1}{4}b^2-2a^3e}{a^2}v^2-\frac{be}{a}v^3$ and by reduction we obtain $v=\frac{\frac{1}{4}b^2-2a^3e-a^2c}{a^2d+abe}$.

We have obtained these two values of v by expressing the root in such a manner as to remove the first and second terms a^2+bv, and the last term e^2v^4; we may also in a similar manner contrive two roots which will remove the first term a^2 and the two last $dv^3+e^2v^4$: this is effected by assuming $a+\frac{d}{2e}v+ev^2$, or $-a+\frac{d}{2e}v+ev^2$ for the root, which will enable us to find two new values of v. If we assume $a+\frac{d}{2e}v+ev^2$ for the root, we have $a^2+bv+cv^2+dv^3+e^2v^4 = a^2+\frac{ad}{e}v+\left(\frac{\frac{1}{4}d^2}{e^2}+2ae\right)v^2+dv^3+e^2v^4$, and therefore $bv+cv^2=\frac{ad}{e}v+\left(\frac{\frac{1}{4}d^2}{e^2}+2ae\right)v^2$, and by division $b+cv = \frac{ad}{e}+\left(\frac{\frac{1}{4}d^2}{e^2}+2ae\right)v$, whence $v=\frac{be^2-ade}{\frac{1}{4}d^2+2ae^3-ce^2}$.

When $-a+\frac{d}{2e}v+ev^2$ is assumed as the root, we proceed in a similar manner, and find $v=\frac{be^2+ade}{\frac{1}{4}d^2-2ae^3-ce^2}$. Thus it appears, that in general we may have six different values of v that will render the formula $a^2+bv+cv^2+dv^3+e^2v^4$ a complete square.

¶ 15. When the biquadratic formula $a^2+bv+e^2v^4$ is to be made a square, there are only two of the six methods explained in the preceding paragraph that can be made use of; that is, we may suppose the root to be either $a+\frac{bv}{a}+ev^2$, or $a+\frac{bv}{a}-ev^2$. But in this case we may successfully apply a root of only two terms, and we shall arrive at two new values of v; we have only to denote the root by $a+ev^2$, or by $a-ev^2$; if we adopt $a+ev^2$ for the root, we have the equation $a^2+bv+e^2v^4=a^2+2aev^2+e^2v^4$, whence bv

$=2aev^2$, and therefore $v=\frac{b}{2ae}$. And if we assume $a-ev^2$ for the root, we shall find by a similar process that $v=-\frac{b}{2ae}$.

The formula $a^2+dv^3+e^2v^4$ may be treated in the same manner; we may assume $a+ev^2$ for the root, from which assumption there arises the equation $a^2+dv^3+e^2v^4=a^2+2aev^2+e^2v^4$; whence by subtraction, $dv^3=2aev^2$, and $dv=2ae$, and $v=\frac{2ae}{d}$. By assuming $a-ev^2$ for the root, we find, after a like operation, that $v=-\frac{2ae}{d}$.

When both b and e are wanting in the formula $a^2+bv+cv^2+dv^3+ev^4$, it is reduced to the cubic formula $a^2+cv^2+dv^3$, which may be made a square in a manner different from any we have mentioned in treating of cubics; in fact, we did not discover it until we were thus far advanced in the present paragraph. The method is this; assume $a+\frac{cv^2}{2a}$ for the root, which squared and equated with the proposed formula, produces this equation, $a^2+cv^2+dv^3=a^2+cv^2+\frac{c^2v^4}{4a^2}$; therefore $dv^3=\frac{c^2v^4}{4a^2}$, and by division $d=\frac{c^2v}{4a^2}$, from which we derive $v=\frac{4a^2d}{c^2}$. The same root will also apply to the biquadratic formula $a^2+cv^2+dv^3+ev^4$.

There are two particular cases of biquadratic formulas having all their terms which are very easily reduced to squares, viz. $a^2+2abv+b^2v^2+dv^3+ev^4$, and, $a^2+bv+c^2v^2+2cev^3+e^2v^4$; the former of these becomes a square when $v=-\frac{d}{c}$, and the latter when

$v=-\frac{a^2}{b}$: these values are too easily discovered to stand in need of any investigation.

¶ 16. That the learner may be able to use with advantage the rules we have given for formulas of the fourth degree, we shall select a few examples which will render their application sufficiently obvious.

Let $4-5v+6v^2-7v^3+8v^4$ be made a square. Here we are to assume $2-rv+sv^2$ for the root, which squared and equated with the proposed formula, gives $4-5v+6v^2-7v^3+8v^4=4-4rv+(r^2+4s)v^2-2rsv^3+s^2v^4$; and assuming $4r=5$, and $r^2+4s=6$, we have $r=\frac{5}{4}$, and $s=\frac{6-r^2}{4}=\frac{71}{64}$; also removing the equals from both sides of the equation, we have $8v^4-7v^3=s^2v^4-2rsv^3$, which divided by v^3 becomes $8v-7=s^2v-2rs$, whence $v=\frac{7-2rs}{8-s^2}=\frac{168}{217}$, which value, substituted instead of v, will render $4-5v+bv^2-7v^3+8v^4$ a complete square.

For another example, let $1+2v+4v^2+8v^3+v^4$ be made a rational square. The first term being a square, we may suppose $1+v+sv^2$ for the root, and we have $1+2v+4v^2+8v^3+v^4=1+2v+v^2+2sv^2+2sv^3+s^2v^4$, and equating the coefficients of v^1, we have $2s+1=4$, and $s=\frac{3}{2}$; also $8v^3+v^4=2sv^3+s^2v^4$, and dividing by v^3, we have $8+v=2s+s^2v$, whence $v=\frac{8-2s}{s^2-1}=4$, and therefore the number 4 substituted for v will make $1+2v+4v^2+8v^3+v^4$, a rational square.

The last term of $1+2v+4v^2+8v^3+v^4$ being a square, we may also assume $s+4v+v^2$ for the root, which squared and equated to the proposed expression gives $1+2v+4v^2+8v^3+v^4=s^2+8sv+(16+2s)v^2+8v^3+v^4$: now let us equate the coefficients of v^2, and we have $16+2s=4$, or $s=-6$, and therefore $1+2v=s^2+8sv=36+48v$, whence $v=\frac{7}{-46}$, which va-

lue of v will also render the proposed expression a square.

Again, both the first and last terms of $1+2v+4v^2+8v^3+v^4$ being squares, we may assume $1+v+v^2$, $1+v-v^2$ for the root. The former of these roots being squared, we have $1+2v+4v^2+8v^3+v^4=1+2v+3v^2+2v^3+v^4$, whence $4v^2+8v^3=3v^2+2v^3$, or $4+8v=3+2v$, and consequently $v=-\frac{1}{6}$; which negative value put for v, will also make the given expression a complete square.

If $1+v-v^2$ be assumed for the root we shall have $1+2v+4v^2+8v^3+v^4=1+2v-v^2=2v^3+v^4$, and therefore $4v^2+8v^3=-v^2-2v^3$, or $4+8v=-1-2v$, from which is found $v=\frac{1}{2}$; which value of v equally fulfils the condition required.

Further, we may also assume $1+4v+v^2$, or $-1+4v+v^2$ for the root. The former of these furnishes us with the equation $1+2v=4v^2+8v^3+v^4=1+8v+18v^2+8v^3+v^4$; and therefore $4v^2+2v=18v^2+8v$, or $4v+2=18v+8$, whence $v=-\frac{3}{7}$; a value which will not fail to render the proposed expression a complete square.

If $-1+4v+v^2$ be supposed the root, we have $1+2v+4v^2+8v^3+v^4=1-8v+14v^2+8v^3+v^4$; and therefore $2v+4v^2=-8v+14v^2$, or $2+4v=-8+14v$, consequently $v=1$; and this value also makes the proposed formula a complete square. We have therefore found these six values of v, viz. 4, $\frac{7}{10}$, $-\frac{1}{6}$, $-\frac{1}{2}$, $-\frac{3}{7}$, and 1; each of which fulfils the condition required.

Let us take the formula $1+20v+4v^4$ for an example. Here we may suppose either $1+10v+2v^2$, or $1+10v-2v^2$, or $1+2v^2$, or $1-2v^2$: in the first case we shall find $v=-\frac{13}{5}$, in the second $\frac{12}{5}$, in the third 5, and in the fourth -5.

Let $1-v^2+v^3-v^4$ be made a square. Assume $1-\frac{1}{2}v^2$ for the root, and therefore $1-v^2+v^3-v^4=1-v^2+\frac{1}{4}v^4$, consequently $v^3-v^4=\frac{1}{4}v^4$, or $1-v=\frac{1}{4}v$, whence $v=\frac{4}{5}$, which substituted for v makes $1-v^2+v^3-v^4=1-\frac{16}{25}-\frac{64}{125}-\frac{256}{625}=\frac{180}{625}=(\frac{13}{25})^2$.

Let $-1+v-v^2+v^4$ be made a square. Here the root to be assumed is $-\frac{1}{2}+v^2$; therefore squaring and equating we have $-1+v-v^2+v^4=\frac{1}{4}-v^2+v^4$, consequently $-1+v=\frac{1}{4}$, and $v=\frac{5}{4}$; a value which admits of an easy verification.

¶ 17. When a biquadratic or a cubic formula cannot be reduced to any of the cases already stated, it may be proper to inquire whether it cannot be resolved into two factors, one of which is a square. Thus, the biquadratic formula $2-4v+v^2+2v^3-v^4$ is not reducible to any of the forms yet explained, nevertheless it is very easy to make it a complete square; we can even assign in one general expression, all the possible values of v that can satisfy the condition required. For, on examination we find that the equation $2-4v+v^2+2v^3-v^4=0$ has two equal roots each unity, and therefore the proposed expression may be expressed by $(1-v)^2\times(2-v^2)$; it is manifest therefore that we have only to make a square of $2-v^2$, which is easily performed in its fullest extent by the methods already exhibited.

¶ 18. When a biquadratic formula includes only the extreme terms and the middle one, if $a+bv^2+cv^4$, it is in general very difficult to manage: there is however one case of this kind which sometimes occurs, and is very easily made a square; I mean the formula $a^2-b^2v^2+c^2v^4$, which will always be a square when we take $v=\frac{a}{b}$, or $\frac{-a}{b}$, or $\frac{b}{c}$, or $\frac{-b}{c}$, four very simple values. Thus the formula $1-9v^2+v^4$ is a square when $v=\frac{1}{3}$, or $-\frac{1}{3}$, or 3, or -3; but the method of discovering these values is too obvious to be insisted on.

¶ 19. When we have by trial or otherwise discovered one value of the unknown quantity that makes a formula of the fourth order a complete square, we may in general find as many others as we please: an easy example will be sufficient to explain this. Let $2-v^4$ be made a square.

Here one value of v is evidently unity; but we wish to obtain another value. For this purpose let us put $v=1+y$, and by this substitution we have $2-v^4=1-4y-6y^2-4y^3-y^4$; and assuming the root $1-2y-5y^2$, we have $1-4y-6y^2-4y^3-y^4=1-4y-6y^2+20y^3+25y^4$, whence $-4y^3-y^4=20y^3+25y^4$, therefore $y=-\frac{12}{13}$, and consequently $v=1+y=1-\frac{12}{13}=\frac{1}{13}$; if therefore we substitute $\frac{1}{13}$ for v, the formula $2-v^4$ will undoubtedly become a complete square. If now we desired more values of v, we should only have to substitute $\frac{1}{13}+y$ for v in the formula $2-v^4$, and proceed as before.

¶ 20. There are yet several other particular methods for equations of the fourth order besides those already mentioned; but we have assigned a sufficient portion of our paper to such equations: we shall therefore conclude our remarks on them at present with observing that there are innumerable formulas of this order that cannot possibly become squares. $2v^4-2v^2+2$ is such a one: whatever value we assign to v, whole or fractional, this formula can never be a complete square. But the most remarkable of these impossible formulas are the following, viz. v^4+1, v^4-1, and $1-v^4$, or which are the same in effect by introducing two unknown quantities, v^4+y^4, and v^4-y^4, neither of which can ever be a square. Also since $v^4-y^4=(v^2+y^2)\times(v^2-y^2)$, it follows that the formulas v^2+y^2, and v^2-y^2, cannot be both squares at the same time, and that neither of the formulas $\frac{v^2+y^2}{v^2-y^2}$, $\frac{v^2-y^2}{v^2+y^2}$ can be a square; these by taking $v=1$, become $\frac{1+y^2}{1-y^2}$, and $\frac{1-y^2}{1+y^2}$, neither of which can ever be a square: lastly, it is easy to demonstrate that $\frac{1+2y^2}{1+y^2}$ never can be a square.

¶ 21. These impossible formulas throw light on some of the most abstruse subjects in mathematical

science; I mean the nature of such fluxionary expressions as $\dot{y}\sqrt{1-y^4}$, $\dot{y}\sqrt{\frac{1+y^2}{1-y^2}}$, and $\dot{y}\sqrt{\frac{1+2y^2}{1+y^2}}$, &c. No mathematician has been able to reduce such expressions into forms of which the fluents may be assignable by finite algebraic quantities, or by circular arcs and logarithms. The reason of this seems to be, that whatever substitution we make for y, or $\dot{y}$, we shall never be able to remove the surd, and obtain an expression in monomials or the form of a rational fraction, because these expressions cannot possibly become rational. Every fluxion of which the fluent is assignable in algebraic terms alone, may be transformed into one or more monomials free from surds, and having only given quantities in the denominators: and every fluxion of which the fluent is assignable by circular arcs and logarithms, may be freed from surds, and reduced to one or more rational fractions, having the variable quantity in the denominators; but $\dot{y}\sqrt{\frac{1+y^2}{1-y^2}}$, and $\dot{y}\sqrt{\frac{1+2y^2}{1+y^2}}$, which are the fluxions of elliptic and hyperbolic arcs, cannot possibly become rational, and therefore cannot have their fluents assigned by circular arcs, and logarithms; in other words, elleptic and hyperbolic arcs cannot possibly be measured by the arcs of the circle and parabola.

¶ 22. *Let $a+bx$ be made a rational cube.*

Assume m for the root, $a+bx=m^3$, from which we obtain $x=\frac{m^3-a}{b}$: this is too easy to require illustration.

¶ 23. *Let $a+bx+cx^2$ be made a rational cube.*

Here it is necessary that the first term should be a cube, in which circumstance our formula becomes $a^3+bx+cx^2$. Let us assume $a+rx$ for the root, which cubed and compared with the proposed expression gives $a^3+bx+cx^2=a^3+3a^2rx+3ar^2x^2+$

r^3x^3 ; and taking $3a^2r=b$, which determines r, we have $cx^2=3ar^2x^2+r^3x^3$, and by division $c=3ar^2+r^3x$, whence $x=\frac{c-3r^2}{r^3}$. This case may be very properly exemplified in the formula $1+x+x^2$; but this we leave to the learner as an exercise.

¶ 24. *Let* $a^3+bx+cx^2+dx^3$ *be made a cube.*

Assuming $a+rx$ for the root, we shall have this equation, $a^3+bx+cx^2+dx^3=a^3+3a^2rx+3ar^2x^2+r^3x^3$; and making $3a^2r=b$, which determines the value of r, we have $3ar^2x^2+r^3x^3=cx^2+dx^3$, which by division becomes $3ar^2+r^3x=c+dx$, and therefore the value of $x=\frac{3ar^2-c}{d-r^3}$.

As an example of this method, let $1-x+x^2-x^3$ be made a rational cube. By assuming $1-rx$ for the root, we have $1-x+x^2-x^3=1-3rx+3r^2x^2-r^3x^3$; and by making $3r=1$. or $r=\frac{1}{3}$, we have $x^2-x^3=3r^2x^2-r^3x^3$, which divided by x^2 becomes $1-x=3r^2-r^3x$, and therefore $x=\frac{1-3r^2}{1-r^3}=\frac{0}{\frac{1}{3}}$, which value being substituted for x, we find $1-x+x^2-x^3=\frac{1000}{2197}=(\frac{10}{13})^3$.

¶ 25. When the last term of a cubic formula is already a cube, as in $a+bx+cx^2+d^3x^3$, we may substitute $\frac{1}{y}$ for x, and the resulting formula multiplied by y^3 will become $d^3+cx+bx^2+a$, which coincides with that resolved in the last paragraph. But we may proceed in a more direct manner by assuming $r+dx$ for the root, which cubed and equated with the proposed formula, gives $a+bx+cx^2+d^3x^3=r^3+3r^2dx+3rd^2x^2+d^3x^3$: now let us make $3rd^2=c$, or $r=\frac{c}{3d^2}$, and we shall have $a+bx=r^3+3r^2dx$, a simple equation, from which we obtain $x=\frac{a-r^3}{3r^2d-b}$. Of this we have an example in the formula x^3-x^2+x

—1. In this example $a=-1$, $b=1$, $c=-1$, $d=1$, and $r=\frac{c}{3d^2}=-\frac{1}{3}$, and consequently $x=\frac{a-r^3}{3r^2d-b}=\frac{13}{9}$, which value of x will make x^3-x^2+x-1 a complete cube.

¶ 26. When both the first and last terms are cubes, as in $a^3+bx+cx^2+d^3x^3$, we make the formula become a cube by a different value of x from those given by the methods explained in the 24th and 25th paragraphs. We may assume $a+dx$ for the root, this cubed and compared with the proposed formula gives $a^3+bx+cx^2+d^3x^3=a^3+3a^2dx+3ad^2x^2+d^3x^3$; whence $bx+cx^2=3a^2dx+3ad^2x^2$, and by division $b+cx=3a^2d+3ad^2x$, from which we obtain $x=\frac{3a^2d-b}{c-3ad^2}$. Example, let $1+2x+4x^2+x^3$ be made a cube. Here $1+x$ is to be assumed as the root, and we have $1+2x+4x^2+x^3=1+3x+3x^2+x^3$, whence $2x+4x^2=3x+3x^2$, or $x^2=x$, and consequently $x=1$, which value being put for x makes $1+2x+4x^2+x^3$ $1+2+4+1=8=2^3$.

Two other values of x may be found that will render $1+2x+4x^2+x^3$ a complete cube, by pursuing the methods taught in the 24th and 25th paragraphs.

¶ 27. When the first and second terms are both wanting, the general cubic formula becomes cx^2+dx^3, which is made a cube with the greatest ease. We have only to assume $cx^2+dx^3=m^3x^3$, which by division becomes $c+dx=m^3x$, from which is had $x=\frac{c}{m^3-d}$, and m may be taken at pleasure.

¶ 28. Sometimes we easily obtain the general value of the unknown quantity, which will make $a+bx+cx^2+dx^3$ a complete cube. This always happens when the equation $a+bx+cx^2+dx^3=0$ has two equal roots. To exemplify this let $2-7x+8x^2-3x^3$ be made a cube.

By putting $2-7x+8x^2-3x^3=0$ we discover that

there are two equal roots each unity, and therefore $2-7x+8x^2-3x^3=(2-3x)\times(1-x)^2$. Now this last is made a cube in a general manner by assuming $(2-3x)\times(1-x)^2=\frac{m^3}{n^3}(1-x)^3$: from this we obtain the simple equation $2-3x=\frac{m^3}{n^3}(1-x)$, whence $x=\frac{2n^3-m^3}{3n^3-m^3}$, in which m and n may be taken at pleasure. If $m=n=1$, we find $x=\frac{1}{2}$, which value substituted for x will not fail to render $2-7x+8x^2-3x^3$ a perfect cube.

¶ 29. When we have by trial or otherwise discovered one value of the unknown quantity that will render a quadratic or cubic formula a perfect cube, we may generally find as many other values as we please by a proper substitution. The method of finding these will be sufficiently understood from the following example: let $7-x^3$ be made a cube. Here we easily find one value of x that fulfils the proposed condition, viz. $x=-1$, for $7-(-1)^3=7+1=8=2^3$; let us therefore put $x=-1+y$, and by substitution $7-x^3=8-3y+3y^2-y^3$, which must be a cube; suppose its root $2-y$, and we have $8-3y+3y^2-y^3=8-12y+6y^2-y^3$, whence $-3y+3y^2=-12y+6y^2$, or $-3+3y=-12+6y$, and therefore $y=3$, and of course $x=3-1=2$, which value in fact fulfils the required condition, for $7-x^3=7-8=-1=(-1)^3$. But by assuming the root differently we may find another value of y from the same equation. Let $2-ry$ denote the root, and we have $8-3y+3y^2-y^3=8-12ry+6r^2y^2-r^3y^3$; which by assuming $12r=3$, or $r=\frac{1}{4}$, becomes $3y^2-y^3=6r^2y^2-r^3y^3$, and dividing by y^2 we have $3-y=6r^2-r^3y$, and therefore $y=\frac{3-6r^2}{1-r^3}=\frac{8}{7}$, whence $x=\frac{5}{7}$; and this value of x makes $7-x^3=7-\frac{125}{27}=\frac{189-125}{27}=\frac{64}{27}=\left(\frac{4}{3}\right)^3$.

¶ 30. But in many instances all our endeavours are ineffectual in such researches; in fact there are innumerable formulas that never can become cubes: among these $1+x^3$, $1-x^3$, x^3-1, are the most singular, or by introducing two unknown quantities x^3+y^3, and x^3-y^3, neither of which can ever become a cube, as may be completely demonstrated.

¶ 31. Having thus briefly run through several of the most common cases of single equalities, we shall say a few words on double and triple equalities, which are of most frequent use in the Diophantine Analysis.

Let us find x such that $ax+b$, and $cx+d$ may be both squares.

Assume $ax+b=z^2$, whence $x=\frac{z^2-b}{a}$, and therefore $cx+d=\frac{cz^2-cb+ad}{a}=\frac{acz^2+a^2d-abc}{a^2}$, which must be a square, and therefore acz^2+a^2d-abc must be a square, and may be treated according to the rules already given for single formulas of the second degree.

Example. To find such a value of x that $2x+3$, and $4x+5$ may be both squares. Assume $2x+3=z^2$, whence $x=\frac{z^2-3}{2}$, and $4x+5=2z^2-1$, which must be a square. By pursuing the methods taught in the preceding paragraphs for formulas of the second degree, we may find as many values of z as we please, and among others 5, 29, 169. If $z=5$, we have $x=\frac{z^2-3}{2}=11$, and $2x+3=22+3=25=5^2$, also $4x+5=44+5=49=7^2$; if $z=29$, we have $x=419$, and if $z=169$, we find $x=14279$. The learner may further exemplify this by finding such values of x as will make $2+x$, and $2-x$ both squares.

¶ 32. *Let us find x such as will render $ax+bx^2$, and $cx+dx^2$ both squares.*

Dividing each of these formulas by x^2, the quotients $\frac{a}{x}+b$, and $\frac{c}{x}+d$ must evidently be squares, which

by putting $y=\frac{1}{x}$ become $ay+b$, and $cy+d$, formulas already considered in the last paragraph. The reader may try as an example, to find x such that $x+x^2$, and $x-x^2$ may both be squares.

¶ 33. *Let* $ax+b$, *and* cx^2+d *be made squares by the same value of* x.

Assume $ax+b=z^2$, whence $x=\frac{z^2-b}{a}$, and $cx^2+d=\frac{c(z^2-b)^2+a^2d}{a^2}$ = a square, and therefore $c(z^2-b)^2+a^2d=cz^4-2bcz^2+a^2d+b^2c$ must be a square. This is to be effected when possible by the rules given for biquadratic formulas.

Example. To find x such that $x+1$, and x^2+1 may both become squares. Put $x+1=z^2$, therefore $x=z^2-1$, and $x^2+1=z^4-2z^2+2$ must be a square, and to have x positive, z must be greater than 1. Since the formula z^4-2z^2+2 is a square when $z=1$, let us put $z=1+y$, and by substitution $z^4-2z^2+2=1+4y^2+4y^3+y^4$: suppose now, according to a remark in the 15th paragraph, that $1+2y^2$ is the root, and we have $1+4y^2+4y^3+y^4=1+4y^2+4y^4$ wherefore $4y^3+y^4=4y^4$, or $4=3y$, whence $y=\frac{4}{3}$, $z=1+\frac{4}{3}=\frac{7}{3}$, and $x=z^2-1=\frac{40}{9}-1=\frac{40}{9}$. Now this value $\frac{40}{9}$ put for x will evidently make $x+1$, and x^2+1 both squares, for $\frac{40}{9}+1=\frac{49}{9}=(\frac{7}{3})^2$, and $x^2+1=\frac{1600}{81}+1=\frac{1681}{81}=(\frac{41}{9})^2$. The reader will no doubt observe, that the methods given in this paragraph applies equally to the formulas $ax+b$, and cx^2+dx+e.

¶ 34. *Let us find* x *such that* ax^2+b, *and* cx^2+d *may be both squares.*

We must begin with satisfying ourselves that each of the proposed formulas may become a square. Then having found by trial or otherwise a value r, which, substituted for x, will render ax^2+b a square, we must put $x=r+y$, and by this substitution the proposed formulas are transformed into $ar^2+b+2ary+ay^2$, and $cr^2+d+2cry+cy^2$. And since, when $x=$

r, $ax^2+b=ar^2+b=$ a known square $=e^2$, our first formula becomes $e^2+2ary+ay^2$; and, putting $cr^2+d=f$, our second becomes $f+2cry+cy^2$. To make $e^2+2ary+ay^2$ a square, imagine $zy-e$ to be the root, and there arises the equation $e^2+2ary+ay^2=e^2-2ezy+z^2y^2$, whence by reduction $y=\frac{2ar+ez}{z^2-a}$. This value of y being substituted for it in $f+2cry+cy^2$, will produce a formula of the fourth degree in terms of z, which may, when possible, be made a square by the rules already taught for such single equalities.

We are by no means to conclude that both equations can become squares with the same value of x, merely because each may be made a square with a different value: the contrary happens frequently; the formulas x^2+1. and x^2-1 may both become squares, but not with the same value of x, for then $(x^2+1)\times(x^2-1)=x^4-1$ would also be a square, which we know to be impossible.

It is easy to see that a similar method may be used to find such a value of x as will render the formulas ax^2+bx+c, and dx^2+ex+f complete squares.

We shall take an example. To find x such that x^2+1, and x^2+5 may both be squares. It is evident that each of these formulas may be made a square separately; therefore we may commence with making x^2+1 a square. Suppose $zx-1$ to be its root, and we shall have the equation $x^2+1=1-2zx+z^2x^2$, whence by reduction $x=\frac{2z}{z^2-1}$, and therefore $x^2+5=\left(\frac{2z}{z^2-1}\right)^2+5=\frac{5z^4-6z^2+5}{(z^2-1)^2}=$ a square, and therefore $5z^4-6z^2+5$ must be a square. The slightest attention is sufficient to discover that when $z=1$, the formula $5z^4-6z^2+5$ becomes a square; but this value of z gives $x=\frac{2}{1-1}$, from which we can draw no conclusion: this value of z therefore does not answer

our purpose, and if, according to what has been taught, we substitute $1+y$ for z, we shall arrive at another biquadratic formula equally deceptive. In fact, this is one of those embarrassing formulas which often mock our ingenuity; nothing, therefore, remains for us to do, but to prove the impossibility of the problem proposed.

We succeed better in attempting to assign x such that x^2+2, and x^2+11 may both be squares, and we find $x=\frac{7}{4}$, we may easily find as many other values as we please.

¶ 35. There is also a particular and useful method of determining x such that a^2x^2+bx+c, and d^2x^2+ex+f may both be squares, which we must not omit. We should first, when necessary, reduce by multiplication or division, the square coefficients of x^2 to an equality, which is sufficiently easy; the operation afterwards will be plain from an example. To find x such that x^2-x+7, and x^2-7x+1 may both be squares. Suppose $x^2-x+7=A^2$, and $x^2-7x+1=B^2$; whence, by subtraction $6x+6=A^2-B^2$, or $(2x+2)\times 3=(A+B)\times(A-B)$: now assume $2x+2=A+B$, $3=A-B$, whence $x+\frac{5}{2}=A$, and $x-\frac{1}{2}=B$. Each of these will give the same value of x; if we use A, we have $x^2-x+7=x^2+5x+\frac{25}{4}$, from which we obtain $x=\frac{1}{8}$, a value which will make both the formulas squares.

¶ 36. *Let us find x such that $ax+b$, $cx+d$, and $ex+f$ may all be squares.*

Put $ax+b=y^2$, whence $x=\frac{y^2-b}{a}$, and this value of x being substituted for it in the remaining formulas $cx+d$, and $ex+f$, they become $\frac{acy^2+a^2d-abc}{a^2}$, and $\frac{aey^2+a^2f-abe}{a^2}$: the question therefore is reduced to the finding of x such that the two quadratic formulas acy^2+a^2d-abc, and aey^2+a^2f-abe may both be squares.

By applying this method to the formulas $1-x$, $2-x$, $8-x$, we find $x=\frac{7}{16}$.

¶ 37. Such are some of the most general methods of resolving formulas involving only one unknown quantity; but by far the most difficult and curious part of my subject yet remains, I mean the solution of Diophantine problems, in which two or more numbers are required. On this inquiry I cannot enter at present, for my paper has already swelled beyond its intended limits, I must, therefore, defer the farther prosecution of this research to another opportunity.

A List of the Subscribers to the first Volume of the Mathematical Correspondent.

Science has civilized man: its noble and generous patrons ought always to be publicly known and gratefully remembered. G. B.

	Copies.
Dr. Samuel Ackerly, New-York,	1
Simon Ackerman, ditto,	1
Samuel Adams, ditto,	1
Francis Adincourt, Lansingburgh, New-York,	2
Robert Adrain, Reading, Pennsylvania,	1
Mr. Agnew, New-Brunswick, New-Jersey,	1
Joshua Alder, New-York,	1
David Allinson, Burlington, New-Jersey,	6
American Philosophical Society, Philadelphia,	1
Dr Alexander Anderson, New-York,	1
Thomas Anderson, Waynesborough, Georgia,	1
John Anthon, New-York,	1
James Baehr, ditto,	1
Henry Bates, ditto,	2
Wheeler Barnes, Burlington, Vermont,	6
Abner Bennedict, Springfield, New-Jersey,	1
Thomas Bennet, New-York,	1
Daniel Berry, ditto,	1
Professor Blackburn, College of William and Mary, Virginia,	1
Drayton Blackwell, New-York,	1
Daniel Blain, Lexington, Virginia,	1
Anthony Bleecker, New-York,	1
Cornelius Bogert, ditto,	1
John G. Bogert, ditto,	1

Robert Boorman, New-York, 1
James Boyd, ditto, 2
John Boyd, ditto, 1
Maunsel Bradhurst, ditto, 1
Joshua Brewer, ditto, 1
Joseph Brooks, ditto, 1
Dr. Joseph Brown, Louisiana, 4
Edmund Brown, New-York, 1
George Bryar, ditto, 1
Rinaldo Burleigh, Plainfield, Connecticut, 4
L. S. Burling, New-York, 1
Everart Bush, ditto, 1
Michael Burnham, ditto, 1
Matthias Burnet, Norwalk, Connecticut, 1
Mr. Butler, Baltimore, 6
Thomas Butler, New-York, 1
Jarvis Buxton, Newbern, North-Carolina, 1
Evan Bynan, Long-Island, New-York, 1
Joseph Caldwell, Chapel Hill, North-Carolina, 1
Patrick Calliley, New-York, 1
Geradus Camp, ditto, 2
Malcolm Campbell, ditto, 1
John Capp, Harrisburg, Pennsylvania, 1
R. Cary, Staten-Island, New-York, 1
James B. Carney, Newbern, North-Carolina, 1
Philip Carney, New-York, 1
George Carouthers, York-Town, Pennsylvania, 1
William Carver, New-York, 1
Capt. Uriah O. Champlin, ditto, 1
E. Childs, ditto, 1
George Chilton, Newark, New-Jersey, 1
Clement C. Clark, New-York, 1
James Clark, ditto, 1
David Clark, ditto, 1
Hon. De Witt Clinton, ditto, 1
Richard Cockerell, Ancaster, Canada, 1
Benjamin Coles, New-York, 1
Capt. B. Collins, ditto, 1
Jonathan Colfax, ditto, 1
Elisha Comstock, ditto, 1
Jacob Concklin, Rockland, New-Jersey, 1
George W. Cook, Newbern, North-Carolina, 1
John Coope, Chester County, Pennsylvania, 1
Amos Corning, New-York, 1
Peter S. Cortilyou, Long-Island, 1
Edward Cotton, Boston, 1
Joseph Cowing, Alexandria, 1
John Craggs, near Richmond, Virginia, 1

1	John D. Craig, Baltimore,	1
2	James Gregan, Brooklyn, New-York,	1
1	Henry Cullen, ditto,	1
1	William Cumberland, ditto,	1
1	Mr. Cutting, ditto,	1
1	G. Davis, Washington, Missisippi Territory,	1
4	James Dean, Montpelier, Vermont,	1
1	Abraham De Baum, New-York,	1
1	Charles De France, Washington, Missisippi Territory,	1
4	Peter Deitrich, New-York,	1
1	Peter Delamar, Philadelphia,	3
1	Charles Denning, New-York,	1
1	J. B. Desese, ditto,	1
1	Simeon De Witt, Albany,	1
6	C. Dickinson, New-York,	1
1	Peter Dobbs, ditto,	1
1	ALEXANDER DOIG, ditto,	1
1	John Dominick, ditto,	1
1	Daniel Dominick, ditto,	1
1	Gad Dumbolton, ditto,	1
2	Duncan Drummond, ditto,	1
1	WILLIAM DUNBAR, Natches, Missisippi Territory,	1
1	Richard Dunn, New-York,	1
1	William Dunscomb, ditto,	1
1	P. S. Dupenceau, Philadelphia,	1
1	John Eastmond, New-York,	1
1	John Eberly, Lancaster, Pennsylvania,	1
1	William Edgar, New-York,	1
1	William Elliot, New-Brunswick, New-Jersey,	1
1	Restore Estlack, Woodstown, New-Jersey,	1
1	Charles Farquhar, New-York,	1
1	W. P. Farrand, Philadelphia,	1
1	Peter Fenton, New-York,	1
1	Thomas G. Fesenden, New-York,	1
1	Rev. Collin Ferguson, Washington College, Maryland,	1
1	Moses Field, New-York,	1
1	Charles Finegan, ditto,	1
1	Walter Folger, Nantucket,	6
1	John Forsithe, York-Town, Pennsylvania,	1
1	Donald Fraser, New-York,	1
1	Mr. Frost, ditto,	1
1	John Garnet, New-Brunswick, New-Jersey,	1
1	Mr. Gautier, New-York,	1
1	William Gibbons, Pennsylvania,	1
1	Edward Giddons, Baltimore,	1
1	Walter Gilbert, Washington, Missisippi Territory,	1
1	John Goldsmith, New-York,	1
1	James Gosman, ditto,	1

Peter M'Gowan, New-York,	1
William Green, ditto,	1
Timothy Greene, ditto,	1
Daniel Grey, ditto,	2
James Grey, Philadelphia,	1
John Griscomb, Burlington New-Jersey,	1
John Gummere, Burlington, New-Jersey,	1
Jacob Halsey, New-York,	1
Daniel Halstead, ditto,	1
Alexander Hamilton, ditto,	1
T. Hamilton, Philadelphia,	1
Thomas Hamersly, New-York,	1
John M. Hanchel, Burlington, New-Jersey,	1
James Hardy, New-York,	1
Jonathan Harned, ditto,	1
Thomas Harrison, ditto,	1
Richard Hatfield, ditto,	1
Seaman Hauxhurst, ditto,	1
John Haviland, ditto,	1
Timothy Hedges, ditto,	1
William Hembil, Philadelphia,	1
John Henry, Baltimore,	1
Adrian B. Hicks, New-York,	1
Dr. Hicks, ditto,	1
Thomas Holiday, Hackensack, New-Jersey,	1
Mr. Hollingsworth, Fredericksburgh, Virginia,	1
James Hornet, New-York,	1
Jesse Hoyt, ditto,	1
James Hughes, Philadelphia,	1
Rev. Thomas P. Irving, Newbern, North-Carolina,	4
Dr. Jackson, New-York,	1
Oliver Jaques, ditto,	1
Sir James Jay, ditto,	1
Richard Jennings, ditto,	1
George Jewison, ditto,	1
John R. Johnson, ditto,	1
Richard Johnson, ditto,	1
John I. Johnson, ditto,	1
William Kean, Newbern, North-Carolina,	1
Jarvis Kenan, New-York,	1
Thomas H. Kennedy, ditto,	1
David King, ditto,	1
Capt. William King, ditto,	1
Isaac Kingsland, ditto,	1
Dr. R. S. Kissam, ditto,	1
Jacob Kitchill, ditto,	1
Henry Kneeland, ditto,	1
John Langden, ditto,	1

Garret Lansing, Albany,	1
John Lawrence, New-York,	1
Dr. Peter Lawrence, ditto,	1
B H. Latrobe, Wilmington, Delaware,	1
Caleb Leach, New-York,	1
Thomas Lefferts, ditto,	1
Barnum Leonard, ditto,	1
Anthony Lispenard, ditto,	1
Enoch Lewis, West-Town, Pennsylvania,	1
John R. Livingston, New-York,	1
I. I L. Lockwood, ditto,	1
George Ludlow, ditto,	1
Gulian Ludlow, ditto,	1
Christopher B Ludlow, ditto,	1
William Ludlow, ditto,	1
James Luckie, ditto,	1
David M'Andrew, ditto,	1
James M'Cormie, Carlisle, Pennsylvania,	1
Dr. Maclean, Princetwon College, New-Jersey,	1
Alexander Macomb, New-York,	1
William Macomb, ditto,	1
Mr. M'Farlane, ditto,	1
James M'Ginness, Harrisburg, Pennsylvania,	1
John M'Gregor, New-York,	1
Dr. MACKAY, London, England,	50
James M'Kean, New-York,	1
Alexander M'Kenzie, ditto,	1
William M'Leland, ditto,	1
Thomas M'Murray, York-Town, Pennsylvania,	1
Robert Macquin, New-York,	1
John Mason, ditto,	1
Joshua Matthews, Baltimore,	1
Thomas Maughan, Quebec,	20
Peter Maverick, New-York,	1
John Mayo, ditto,	1
Benjamin Mayo, ditto,	1
John Merrick, Hallowell, District of Maine,	1
Dr. Edward Miller, New-York,	1
Samuel B. Mills, ditto,	1
Samuel Moody, Hollowell, District of Maine,	1
Samuel Moore, Philadelphia,	1
Samuel Moore, Salisbury, Connecticut,	1
Samuel Moor, New-York,	1
Joseph Moor, ditto,	1
Warder Morris, Reading, Pennsylvania,	1
Sylvester Morris, New-York,	1
General Morton, ditto,	1
George Nagle, Reading, Pennsylvania,	1

Melatiah Nash, New-York,	2
I. E. Nolon, Washington, Mississippi Territory,	1
Frederick North, Newbern, North Carolina,	1
Frederick North, New-York,	1
Mrs B Norton, ditto,	1
B. Oblenus, ditto,	1
Rev. Dr. O'Brien, ditto,	1
Benjamin Ogden, ditto,	1
H Onderdonk, ditto,	1
Mr Osborne, ditto,	1
Isaac Parker, ditto,	1
Prof. Patterson, Philadelphia,	1
Walter Patterson, New-York,	1
Albert Picket, ditto,	6
Mr. Plum, New-Brunswick, New-Jersey,	1
John R. Pluymert, New-York,	1
Mr. Pool, New-Brunswick, New-Jersey,	1
Major Popham, New-York,	1
Archibald Post, ditto,	1
Hon J. B. Prevost, New-Orleans,	1
Samuel Purdy, New-York,	6
R Quackenbush, ditto,	1
William S. Radcliff, ditto,	1
John Rathbone, ditto,	1
James Redmond, ditto,	1
Register of the Treasury, Washington City,	2
David Reins, New-York,	1
Mr. Reynolds, Bookseller, ditto,	1
Charles Richards, Reading, Pennsylvania,	1
William S. Richardson, New-York,	1
John Riley, Philadelphia,	1
Major Rivardi, ditto,	1
James Robb, ditto,	1
Walter Robbins, New-York,	1
George Robertson, ditto,	1
Charles Roberts, Pennsylvania,	1
Seth Robson, Savannah, Georgia,	1
J. Romaine, New-York,	1
James Ross, ditto,	1
Fredrick Roorbach, ditto,	1
Mr. Rudd, ditto,	1
James Russel, Dauphin County, Pennsylvania,	1
William Salter, Woodbridge, New-Jersey,	1
David Sanford, Cincinnati, Ohio,	1
Edward Schieffelin, New-York,	1
William Scott, A. M. ditto,	1
Solomon Seixas, ditto,	1
Capt. Nathaniel Shaler, ditto,	1

Samuel Shapter, New-York, 1
Alpheus Sherman, ditto, 1
B. Shays, ditto, 1
John Sidel, ditto, 1
Lewis Simond, ditto, 1
Thomas J. Singleton, Newbern, North-Carolina, 1
Hermanus Skaats, New-York, 1
Mr Smith, New-Orleans, 1
J. B. Smith, Philadelphia, 1
Josiah P. Smith, Mount Bethel, South-Carolina, 1
Seth Smith, Baltimore, 1
William Smith, New-York, 1
Henry Smith, ditto, 1
Francis Smith, ditto, 1
William S. Smith, ditto, 1
J. Scott Smith, ditto, 1
John Smithis, Philadelphia, 1
Horatio G. Spafford, Chatham, New-York, 1
Peter Spangler, York-Town, Pennsylvania, 1
John Spayd, Reading, ditto, 1
Elias Stansbury, New-York, 1
William Stansbury, ditto, 1
Marcus C. Stephens, Newbern, North-Carolina, 1
Matthew Stevenson, Lancaster, Pennsylvania, 1
John Stevens, New-York, 1
William Stewardson, New-York, Virginia, 1
James T Stuart, New-York, 1
Alexander Stuart, ditto, 1
John Stewart, ditto, 1
Robert A. Stewart, ditto, 1
Samuel Swartwout, ditto, 1
Robert Swartwout, ditto, 6
Henry Swartwout, ditto, 6
Richard Tagart, ditto, 6
John Tait, ditto, 1
Jacob Tallman, ditto, 1
Hon. John L. Taylor, Newbern, North-Carolina, 1
James Temple, New-York, 1
Thomas H Thomas, ditto, 1
Mr. Tice, ditto, 1
Robert Tollerton, ditto, 1
Robert Townsend, ditto, 1
Farming Tracy, Canterbury, Connecticut, 1
S Tredwell, New-York, 1
James Turner, Lexington, Kentucky, 1
Budd Tuttle, Burlington, Vermont, 1
James Tuttle, New-York, 1
Cornelius Van Cliff, ditto, 1

Peter Van Steenbergh, New-York, 1
Richard Varrick, ditto, 1
Benjamin Vaughan, Hallowell, District of Maine, 1
Charles Vaughan, ditto, 1
John Vaughan, Philadelphia, 1
Gulian C. Verplank, New-York, 1
Andrew Verveeler, ditto, 1
Alexander Walsh, ditto, 1
Dr Walters, ditto, 1
Henry Waring, ditto, 1
Joseph Watkins, ditto, 1
George Wear, ditto, 1
Thomas Wear, ditto, 1
William Weatheril, ditto, 4
Professor Webber, Harvard College, Massachusetts, 1
Peter Welbone, New-York, 1
John E. West, ditto, 1
Archibald Whaits, ditto, 1
Ebenezar R. White, Danbury, Connecticut, 1
Thomas Whittaker, York-Town, Pennsylvania, 1
Richard Wilkinson, New-York, 1
David Williamson, ditto, 1
George J Williams, Washington, Missisippi Territory, 1
James Williams, Petersburg, Virginia, 1
Peter Williams, New-York, 1
Thomas Wills, ditto, 1
John Wilson, Boston, 1
Joshua Wingate, Hallowell, District of Maine, 1
Jacob Winooski, Green Mountains, Vermont, 1
James C. Winter, New-Orleans, 1
Benjamin Woods, Newbern, North-Carolina, 1
Darius Yankee, Bunker's Hill, 1
T. Youle, Portland, District of Maine, 1
Abner Young, New-Haven, Connecticut, 1
N. Young, near Lexington, Kentucky, 6

THE END OF VOL. I.